Optimal Mechanism Design with Speculation and Resale*

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Abstract

In this paper, we examine the optimal mechanism design problem of selling an indivisible object to one regular buyer and one speculator, where inter-buyer resale cannot be prohibited. The resale market is modeled as a stochastic ultimatum bargaining game between the two buyers. We fully characterize the optimal mechanism under general conditions. Surprisingly, in the optimal mechanism, the seller never allocates the object to the regular buyer regardless of his bargaining power in the resale market. The seller sells only to the speculator, and reveals no additional information to the resale market. The possibility of resale causes the seller to sometimes hold back the object, which under our setup is never optimal if resale is prohibited. We find that the seller's revenue is increasing in the speculator's bargaining power in the resale market. When the speculator has full bargaining power, Myerson's optimal revenue is achieved. When the speculator has no bargaining power, a conditional efficient mechanism prevails. Extension to the case of one speculator and many regular buyers is also discussed.

Keywords: Auctions, Mechanism Design, Resale, Speculation, Bargaining power

JEL Classifications: C72, D44, D82, D83, L12

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1 Introduction

In the traditional mechanism design literature, buyers cannot resell the objects they just won. For example, in his seminal paper, Myerson [10] characterizes the optimal mechanism of selling an indivisible object to many privately informed buyers. In that mechanism, the buyer with the highest virtual valuation is awarded the object, providing that it is higher than the seller's reservation value. However, the Myerson allocation may be *ex-post* inefficient among the buyers if the buyers are asymmetric, as the highest virtual valuation buyer may not have the highest valuation. Therefore, buyers may be able to benefit from trading among themselves. Thus, when this kind of inter-buyer resale cannot be prohibited, buyers will engage in resale upon the Myerson allocation. Given this, the final allocation of the object rolls away from the Myerson allocation, providing different incentives for the buyers in the mechanism. As such, the Myerson optimal revenue may not be achievable.

Many researchers have started to address this issue of resale in auctions and optimal mechanisms. These include Ausubel and Cramton [1], Calzolari and Pavan [2], Cheng and Tan [3], Garratt and Troger [4], Hafalir and Krishna [5], Haile [6], Virag [13], and Zheng [14], all of which we will discuss in this introduction. Markedly, Zheng [14] demonstrates that even if the seller cannot prohibit resale, she can still achieve the Myerson revenue under certain resale rules. He constructs a mechanism involving many rounds of resales, with the winner of each round reselling the object to the rest of the buyers, leading to the Myerson allocation in the end. The Revenue Equivalence Theorem implies that this mechanism generates the same revenue as the Myerson revenue, which is the upper bound revenue among all feasible mechanisms. (Note that the seller cannot earn more revenue by allowing for resales from the Revenue Equivalence Theorem.). As a result, Zheng's mechanism is optimal among all mechanisms with resale. One key feature in Zheng's mechanism is that the winner in each round has full bargaining power, dispensing a mechanism that is optimal for himself. In the case where the winner of each round has less than full bargaining power, Myeron's revenue may no longer be achievable.[1] In this case, Zheng's construction does not apply. We characterize the optimal mechanism when the winner has less than full bargaining power.

Our approach is to analyze the buyers' incentive compatibility constraints and participation constraints directly. We adopt the framework of Garratt and Troger [4] in our analysis. Even though the framework is used by them to study the equilibrium behavior in certain auctions with resale, we find it suitable for examining the optimal mechanism design problem as well. In the model, in addition to the seller, there are two buyers: one regular buyer and one speculator, who can engage in resale activities. The seller cannot control what happens in the resale market.

Departing from Zheng's and Garratt and Troger's assumption that the winner has full bargaining power, we model the resale market as a stochastic ultimatum bargaining game between the two buyers. With certain probability, the winner is picked as the proposer in the ultimatum bargaining game, and with the rest of the probability the loser is picked as the proposer. These probabilities can vary depending on who initially wins the object, and serve as the bargaining powers of the respective buyers.

[1] As we show in this paper, when resale is permitted, the Myerson revenue can be achieved when and only when the winner has full bargaining power.

In our analysis, the seller faces a mechanism design problem with hidden information, hidden actions and multiple agents, since a buyer's valuation is his private information and the buyers' actions in the resale market are not dictated by the seller. Myerson [11] establishes the revelation principle and formulates mechanisms under this general setting, even though explicitly characterizing the optimal mechanism is nontrivial.[2]

We are able to explicitly characterize the optimal mechanism under general conditions. The most striking result is that the seller never allocates the object to the regular buyer directly in the initial market even when he has bargaining power in the resell market. The seller allocates the object to the speculator if his resale augmented virtual valuation is higher than the seller's reservation value (which is normalized to zero). The seller charges the speculator an amount equal to his expected benefit from the trading, leaving him with zero total expected surplus. Although the seller never allocates the object to the regular buyer directly, she nevertheless demands some payment from the regular buyer if she decides not to retain the object. More importantly, the seller reveals no private information to the resale market; the buyers only know who just won the object when entering the resale market.

It turns out that the buyers' bargaining powers in the resale market determine crucially the revenue the seller can optimally achieve. In general, the seller's maximal revenue with resale is less than the maximal revenue with no resale (i.e., the Myerson revenue), since the seller has more controlling power in the case of no resale. In this paper, we show that the Myerson revenue can be achieved only when the speculator has full bargaining power. If the regular buyer has full bargaining power, a conditional efficient allocation is optimal. In this conditional efficient allocation, the allocation among the buyers is always efficient, but the seller may retain the object inefficiently. In fact, the seller's revenue is an increasing function of the speculator's bargaining power. When the regular buyer's augmented virtual valuation is always greater than the seller's reservation value, the seller's revenue is a weighted average of the Myerson revenue and the fully efficient allocation revenue, with weights equal to the players' respective bargaining powers.

Note that the speculator's virtual valuation is always greater than the seller's reservation value in our model, and therefore the seller never retains the object in the Myerson allocation. But with resale, it may be optimal for the seller to retain the object under some circumstances. As an implication, resale induces a more efficient allocation among the buyers, but at the same time introduces a new source of inefficiency. Therefore, whether resale can improve the overall efficiency in the optimal mechanism is ambiguous.

The literature on auctions with resale has provided us with significant insights. Hafalir and Krishna [5], for example, examine the first and second price auctions with possible resale. There are two players, and either the winner or the loser in the auction has the chance to make a take-it-or-leave-it offer to the other. They find that the two players, although asymmetric, win with equal probability in the auctions. They also find that first price auctions generate more revenue than second price auctions. Later, Virag [13] extends their analysis to the case of many players

[2] In the literature of optimal incentive contracts, McAfee and McMillian [8], Laffont and Tirole [7], and McAfee and McMillan [9] characterize the optimal contract for a principal facing agents with privately known abilities, unobservable efforts, but observable outputs. In our model, the seller has no control of the resale market except deciding on how much information to reveal.

who can be classified into two groups: weak and strong. He shows that with more than two players, strong players win more often than weak players. Recently, Cheng and Tan [3] study the asymmetric common-value auctions and apply the results to the revenue ranking in independent private value auctions with resale. In doing so, they generalize the analysis in Hafalir and Krishna [5] by relaxing their regularity assumption and find that the revenue ranking in Hafalir and Krishna [5] could sometimes be reversed. Given resale opportunities, the issue of speculators has also been studied. Garratt and Troger [4] consider the first and second price auctions with many symmetric bidders and an additional pure speculator whose valuation is commonly known to be zero. In the resale market, the winner may use a standard auction with an optimal reserve price as well as an optimal mechanism to resell the object. They find that the speculator can play an active role in the equilibrium.[3]

In this paper, we use a mechanism design approach to characterize the optimal mechanism with resale. In the process, we address one important issue that is not addressed in any of the above papers, that is, how should the seller control the information revelation to the resale market? The common assumption in the literature is that only the transaction price (i.e., the highest bid in a first price auction and the second highest bid in a second price auction) is announced by the seller. The question remains whether this is optimal for the seller to do so. In theory, the seller can have many different options. She can conceal all the information, reveal all the information, reveal the information stochastically or partially, etc. Obviously, it is almost impossible to formulate all possible announcement rules one by one. Our paper takes one step further by considering the optimal rule for information revelation. In the optimal mechanism we constructed, concealing all the information is the rule.

Our paper is closest to Calzolari and Pavan [2], who consider the issue of information transmission to resale market in the optimal mechanism. They mainly focus on the case of reselling the object to a third party. In the case of inter-bidder resale, they assert that any deterministic mechanism cannot be optimal. With two bidders and two-point valuation distributions, they provide a characterization of the optimal mechanism in the case where one of the bidders has full bargaining power (in their online appendix). In our model, the regular buyer has a continuous valuation distribution, while the speculator has a commonly known valuation. Furthermore, we allow the buyers to have medium levels of bargaining powers. Because of these medium levels of bargaining powers and the continuous distribution of one buyer's valuation, we use a different method to characterize the optimal mechanism. One result that we obtain is that the seller's revenue is an increasing function of the bargaining power of the initial winner (who turns out to be the speculator).

Our paper is also related to Ausubel and Cramton [1], who characterize the optimal mechanism when the resale market is perfect in the sense that any inefficiency will be corrected in the resale market. Given such a perfect resale market, they find that the seller should induce an efficient allocation directly in the initial market. However, the question remains how a perfect resale market can be constructed under asymmetric information.[4] Our analysis supports the optimality of the

[3] In all of these papers, resale arises because of the inefficiencies of the auction allocations. This is also the case that we will focus on in this paper. Haile [6] considers a different source of resale: after the auctions, new information becomes available and it alters the buyers' valuations.

[4] Myerson and Satterthwaite [12] show that inefficiency cannot necessarily be corrected in the secondary market.

initial efficient allocation under certain conditions. If the initial loser has full bargaining power in the resale market, and the regular buyer's virtual valuation is always greater than the seller's reservation value, then it is optimal to allocate the object efficiently in the initial market.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we analyze the model and establish some incentive compatibility conditions for the resale market. In Section 4, we analyze the initial market and establish some incentive compatibility conditions for the entire game. In Section 5, we characterize the optimal mechanism for the seller. In Section 6, we conclude.

2 The Model

One seller (she) with one indivisible object faces two buyers. Buyer 1 has a commonly known valuation $v_1 \geq 0$ for the object. Buyer 2's valuation, v_2, is his private information. Assume that v_2 follows a distribution with *c.d.f.* $F(\cdot)$, *p.d.f.* $f(\cdot)$, and support $[a, b]$. We call buyer 1 the *speculator* and buyer 2 the *regular buyer*.[5]

Assume that the hazard rate, $\frac{f(v_2)}{1-F(v_2)}$, is increasing; this is a common assumption to simplify the characterization of the optimal mechanism. Let $J_1(v_1) = v_1$ and $J_2(v_2) = v_2 - \frac{1-F(v_2)}{f(v_2)}$ denote buyer 1 and buyer 2's virtual valuation functions, respectively. Note that buyer 1's valuation is also his virtual valuation, since it involves no uncertainty. In this paper, without loss of generality, we normalize the seller's reservation value of the object to zero.

When the seller has full controlling power and can prohibit resales among the buyers, Myerson's optimal auction yields the highest revenue for the seller. In that optimal auction, the seller should allocate the object to the buyer with the higher nonnegative virtual valuation. Since the speculator's virtual valuation is nonnegative in our model, the seller does not retain the object in Myerson's optimal auction. Furthermore, if buyer 2's virtual valuation is always greater than or equal to buyer 1's virtual valuation, i.e. $J_2(a) \geq v_1$, then the seller should always allocate the object to buyer 2. If buyer 2's virtual valuation is always less than buyer 1's valuation, i.e., $J_2(b) \leq v_1$, then the seller should always allocate the object to buyer 1. In the former case, $J_2(a) \geq v_1$ implies that $v_2 \geq v_1$, $\forall v_2 \in [a, b]$. In the latter case, $J_2(b) \leq v_1$ implies that $v_2 \leq v_1$, $\forall v_2 \in [a, b]$. In these two cases, since the buyer with the higher valuation receives the object, resales (if allowed) will not happen, and the allocations are efficient.

The above observation leads to the following proposition: given those conditions above, there will be no resales given the Myerson allocations, and thus the Myerson revenue (i.e., the maximum revenue) is achieved even if resale is allowed.

Proposition 1 *Suppose that resales between buyers are allowed. If $J_2(a) \geq v_1$ or $J_2(b) \leq v_1$, then the seller can achieve the highest revenue by implementing the Myerson allocation: always assign*

[5] In Garratt and Troger [4], $v_1 = 0$. When $v_1 > 0$, we can interpret the speculator as a dealer with valuation equal to the expected market value of the object.

the object to buyer 2 if $J_2(a) \geq v_1$, and always assign the object to buyer 1 if $J_2(b) \leq v_1$. The allocation is ex post efficient and no resale between the two buyers occurs.

In what follows, we will assume that buyer 1's valuation lies inside the range of buyer 2's virtual valuations.

Assumption 1 $J_2(a) < v_1 < J_2(b)$.

When Assumption 1 holds, the Myerson allocation is sometimes ex post inefficient, as buyer 1's valuation is sometimes higher than buyer 2's virtual valuation but lower than buyer 2's valuation. In this case, there is strictly positive incentive for buyer 1 to resell the object to buyer 2. The objective of this paper is to characterize the optimal mechanism when this kind of resales are allowed (or equivalently, when resales cannot be prohibited by the seller).

Formally, we model this sale with possible resale situation as follows. There are two markets: the initial market and the resale market. The resale market is modeled as a stochastic ultimatum bargaining game and nature randomly picks a buyer as the proposer with certain probability. We allow the probability to depend on who owns the object when entering the resale market (i.e., who won the object in the initial market). To be more concrete, we assume that when the speculator owns the object, with probability λ_1, the speculator is picked to propose a take-it-or-leave-it offer to the regular buyer, and the regular buyer chooses either to accept or reject the offer; with probability $1-\lambda_1$, the regular buyer is picked to propose and things occur similarly. When the regular buyer owns the object, with probability λ_2, the speculator is picked, and with probability $1-\lambda_2$, the regular buyer is picked. Transaction takes place if the proposed offer is accepted. There are no further rounds of bargaining if the proposed offer is rejected. Note that λ_1 and λ_2 capture the speculator's bargaining powers in each situation.

In the initial market, the seller designs a mechanism to sell the object. If the seller decides to retain the object, the game ends; otherwise, the two buyers enter the resale market. The seller has no control over the resale market, i.e., the structure of the resale market is exogenously given and the seller cannot force the buyers to act in any way in the resale market.[6] In the initial market, the seller can decide on object allocation and monetary transfers. In addition, the seller can decide on how to reveal the buyers' information, in an attempt to influence the buyers' actions in the resale market.

In this dynamic mechanism design problem, the seller plays an active role in information revelation. After the seller sees the information (i.e., reports) from the buyers, she can decide on what information to reveal to the buyers. This information could affect the prices in the resale market, and therefore could affect the buyers' behaviors in the initial market. This effect is not present in a static mechanism design problem. In this paper, we assume that the seller has full control over this revelation of information, which is costless to her.

[6]If the seller has full control of the buyers' actions in the resale market, then we are back to the situation of Myerson [10].

The seller in our model needs to design an optimal mechanism taking into consideration the buyers' resale behaviors. The buyers' behaviors in the resale market are not contractible by the seller. This is a mechanism design problem with hidden information, hidden actions and multiple agents. The challenge is not only the seller deciding the object allocations and monetary transfers, but also controlling the information revealing to the resale market.

We make use of the revelation principle in Myerson [11] throughout our analysis and restrict our search of the optimal mechanism to the following direct mechanisms without loss of generality. In the direct mechanisms, buyer 2 first announces his valuation $\tilde{v}_2$ confidentially to the seller, then with probability $x_1(\tilde{v}_2)$ and $x_2(\tilde{v}_2)$, the seller allocates the object to buyer 1 and buyer 2, respectively.[7] In addition, conditional on $\tilde{v}_2$, the seller sends buyers confidential recommendations on what actions to take in the resale market.

The recommendations depend on which buyer wins and which buyer makes the take-it-or-leave-it offer in the resale market. There are four different situations, indexed by who wins and who is picked to make the offer in the resale market.

Case 11: Buyer 1 wins and buyer 1 is picked to make the offer. The seller recommends buyer 1's price offer $p_{11} \in \mathbb{R}$, and buyer 2's acceptance function conditional on buyer 1's price offer $A_{11}(p) \in \{Accept,\ Reject\}$.

Case 12: Buyer 1 wins and buyer 2 is picked to make the offer. The seller recommends buyer 2's price offer $p_{12} \in \mathbb{R}$ and buyer 1's acceptance function conditional on buyer 2's price offer $A_{12}(p) \in \{Accept,\ Reject\}$.

Case 22: Buyer 2 wins and buyer 2 is picked to make the offer. The seller recommends buyer 2's price offer $p_{22} \in \mathbb{R}$, and buyer 1's acceptance function conditional on buyer 2's price offer $A_{22}(p) \in \{Accept,\ Reject\}$.

Case 21: Buyer 2 wins and buyer 1 is picked to make the offer. The seller recommends buyer 1's price offer $p_{21} \in \mathbb{R}$, and buyer 2's acceptance function conditional on buyer 1's price offer $A_{21}(p) \in \{Accept,\ Reject\}$.

Note that the seller needs to recommend an acceptance function $A_{ij}(p)$ as the offer proposer may not follow the seller's price recommendation. Also, all these recommendations p_{ij} and $A_{ij}(p)$ depend on buyer 2's report $\tilde{v}_2$ as the seller makes them after seeing the report, and thus buyer 1 may be able to infer $\tilde{v}_2$ from the recommendations. Throughout the rest of the paper, we will put $\tilde{v}_2$ in the argument as well for notational clarity; i.e., when $\tilde{v}_2$ is reported, buyer i wins and buyer j is picked to make the offer in the resale market, the seller recommends buyer j to make a price offer of $p_{ij}(\tilde{v}_2)$ and the other buyer accepts or rejects according to $A_{ij}(p; \tilde{v}_2)$, where $i = 1, 2,\ j = 1, 2$.

In our model, we assume that the seller sends out the (confidential) recommendations after the object allocation is revealed in the initial market, but before the resale market is open. Specifically, the recommendations are received by the buyers before they learn who is picked to make the offer in the resale market. Therefore, when buyer 1 wins, he privately learns of the recommendations $p_{11}(\tilde{v}_2)$ and $A_{12}(p; \tilde{v}_2)$; the former is the recommendation in case he is picked as the proposer, and the latter

[7] Since buyer 1 does not have any private information, he has no need to report.

is the recommendation in case he is not the proposer. Meanwhile, buyer 2 privately learns of the recommendations $A_{11}(p;\tilde{v}_2)$ and $p_{12}(\tilde{v}_2)$. Likewise, when buyer 2 wins, buyer 1 privately learns of the recommendations $A_{22}(p;\tilde{v}_2)$ and $p_{21}(\tilde{v}_2)$, and buyer 2 privately learns of the recommendations $p_{22}(\tilde{v}_2)$ and $A_{21}(p;\tilde{v}_2)$. It is assumed this way because the stochastic bargaining process is merely a way to model the buyers' bargaining power.[8] Because the recommendations contain information about buyer 2's private report of his valuation (which becomes accurate in equilibrium), buyer 1 can update his belief about buyer 2's valuation upon receiving the recommendations.

In direct mechanisms, the monetary transfers $t_1(\tilde{v}_2)$ and $t_2(\tilde{v}_2)$ from buyer 1 and 2, respectively, to the seller are collected privately at the very end after the resale market is closed. In indirect mechanisms, the money transfers can be collected anytime. If the buyers learn about the money transfers before the resale market is open, then the buyers can update their beliefs. However, such indirect mechanisms have counterparts in direct mechanisms; any information revealed by the money transfers can be conveyed by the recommendations as well. Of course, if a buyer's monetary transfer does not depend on the other buyer's valuation, then it does not need to be collected at the end, since the transfer does not reveal any private information.

The seller maximizes her revenue by selecting the allocation rules, $x_1(\tilde{v}_2)$ and $x_2(\tilde{v}_2)$, recommendations, $p_{11}(\tilde{v}_2)$, $p_{12}(\tilde{v}_2)$, $p_{21}(\tilde{v}_2)$, $p_{22}(\tilde{v}_2)$, $A_{11}(p;\tilde{v}_2)$, $A_{12}(p;\tilde{v}_2)$, $A_{21}(p;\tilde{v}_2)$ and $A_{22}(p;\tilde{v}_2)$, and monetary transfers, $t_1(\tilde{v}_2)$ and $t_2(\tilde{v}_2)$, subject to the feasibility constraints. Since there is only one object to be allocated, we have

$$x_1(v_2) + x_2(v_2) \leq 1, \ \forall v_2. \tag{1}$$

Note that only buyer 2 has private information. The incentive compatibility constraint for buyer 1 (IC_1^R) is that he will follow the recommendation in the resale market, given that buyer 2 truthfully reports his valuation in the initial market and follows the recommendations in the resale market.[9] The incentive compatibility constraint for buyer 2 is that he will report his valuation truthfully in the initial market and follow the recommendations in the resale market, given that buyer 1 follows the recommendations in the resale market. We break up buyer 2's incentive compatibility constraints into two parts. The first part (IC_2^R) is that, if buyer 2 has truthfully reported his valuation in the initial market, it is optimal for him to follow the seller's recommendation in the resale market. The second part (IC_2^I) is that, buyer 2 will truthfully report his valuation in the initial market given that he will behave optimally in the resale market. The participation constraints for the buyers (PC_1 and PC_2) require that participating in the mechanism is better than their outside options, which are normalized to zero here.

To summarize, the mechanism design problem for the seller is

$$\max \ R = \int_a^b t_1(v_2)dF(v_2) + \int_a^b t_2(v_2)dF(v_2)$$

$$\text{subject to: } IC_1^R,\ IC_2^R,\ IC_2^I,\ PC_1,\ PC_2, \text{ and (1).}$$

[8] We can also assume that the seller cannot make recommendations conditioning on who wins. This is discussed in the conclusion.

[9] Buyer 1 has no incentive compatibility constraint in the initial market, since he does not have any private information.

In the following sections, we will examine these constraints one by one, starting backward from the resale market.

3 The Resale Market

3.1 The On-Equilibrium-Path Continuation Game: Establishing IC_1^R and IC_2^R

We first examine an on-equilibrium-path continuation game in which buyer 2 has truthfully reported his true valuation v_2 in the initial market. The incentive compatibility constraints require that both buyers should follow the seller's recommendations in the resale market. The following lemma regarding the buyers' optimal decisions is straightforward.

Lemma 1 *The acceptance recommendations are incentive compatible for the buyers if and only if:*

$$A_{11}(p; v_2) = \begin{cases} Accept, & if\ p \leq v_2; \\ Reject, & if\ p > v_2; \end{cases} \tag{2}$$

$$A_{12}(p; v_2) = \begin{cases} Accept, & if\ p \geq v_1; \\ Reject, & if\ p < v_1; \end{cases} \tag{3}$$

$$A_{22}(p; v_2) = \begin{cases} Accept, & if\ p \leq v_1; \\ Reject, & if\ p > v_1; \end{cases} \tag{4}$$

$$A_{21}(p; v_2) = \begin{cases} Accept, & if\ p \geq v_2; \\ Reject, & if\ p < v_2. \end{cases} \tag{5}$$

When a buyer is making the acceptance decision in the resale market, he only needs to compare his own valuation with the price offer from the other buyer. For example, in **Case 11**, buyer 2 receives the price offer p from buyer 1, and obviously buyer 2 will accept it if and only if p is no higher than his valuation v_2, and reject it otherwise. Buyers will not follow any other recommendations. One important observation that we will make use of in later analysis is that buyer 1's acceptance recommendations $A_{12}(p; v_2)$ and $A_{22}(p; v_2)$ do not depend on v_2. Thus, upon seeing those acceptance recommendations, buyer 1's belief about buyer 2's valuation does not change.

The recommended price offers are much more complicated to determine, and we examine them case by case.

Case 11: In this case, buyer 1 wins and is picked to make the offer. Buyer 1 believes that buyer 2 truthfully reports his type and follows the acceptance recommendations; i.e., $\tilde{v}_2 = v_2$ and buyer 2 will follow the seller's recommendation about the acceptance decision according to Equation (2). The information buyer 1 has when determining the offering price is that he won and received recommendations $p_{11}(v_2) = p_{11}^*$ and $A_{12}(p; v_2) = A_{12}^*(p)$. Thus, buyer 1 chooses his price offer $\tilde{p}$ to maximize his payoff:

$$\begin{aligned}
\max_{\tilde{p}} \quad & v_1 Prob\{v_2 < \tilde{p} | buyer\ 1\ wins,\ p_{11}(v_2) = p_{11}^* \ and\ A_{12}(p; v_2) = A_{12}^*(p)\} \\
+ \quad & \tilde{p} Prob\{v_2 \geq \tilde{p} | buyer\ 1\ wins,\ p_{11}(v_2) = p_{11}^* \ and\ A_{12}(p; v_2) = A_{12}^*(p)\} \\
= \max_{\tilde{p}} \quad & v_1 Prob\{v_2 < \tilde{p} | buyer\ 1\ wins,\ p_{11}(v_2) = p_{11}^*\} \\
+ \quad & \tilde{p} Prob\{v_2 \geq \tilde{p} | buyer\ 1\ wins,\ p_{11}(v_2) = p_{11}^*\} \qquad (6)
\end{aligned}$$

The above equality follows from the fact that $A_{12}(p; v_2)$ does not depend on v_2. Buyer 1's belief about buyer 2's type depends crucially on the recommendation function $p_{11}(v_2)$. If this function is a one-to-one mapping, the recommendation fully reveals buyer 2's private information. In contrast, if this function is a constant, the recommendation will not alter buyer 1's belief at all. Let

$$G_{11}(v_2) = F(v_2 | buyer\ 1\ wins, p_{11}(v_2) = p_{11}^*),$$

and

$$g_{11}(v_2) = f(v_2 | buyer\ 1\ wins, p_{11}(v_2) = p_{11}^*).$$

Then the above maximization problem can be written as

$$\max_{\tilde{p}} \Pi_1 = v_1 G_{11}(\tilde{p}) + \tilde{p}\left[1 - G_{11}(\tilde{p})\right]. \qquad (7)$$

The incentive compatibility constraint then implies that it is optimal for buyer 1 to follow the seller's recommendation. This is summarized in the following lemma.

Lemma 2 *In **Case 11**, the price offer is incentive compatible if and only if*

$$p_{11}(v_2) = argmax_{\tilde{p}}\{v_1 G_{11}(\tilde{p}) + \tilde{p}[1 - G_{11}(\tilde{p})]\},\ \forall v_2.$$

The induced outcome is that trade occurs at price $p_{11}(v_2)$ if $p_{11}(v_2) \leq v_2$ and not occur if $p_{11}(v_2) > v_2$.

Note that the price $p_{11}(v_2)$ is always no less than v_1. This is because a person will never ask a price lower than his valuation to sell an object. If the first order approach to the maximization problem in (7) is valid, its FOC gives us:

$$\begin{aligned}
\frac{d\Pi_{11}}{d\tilde{p}} &= v_1 g_{11}(\tilde{p}) - \tilde{p} g_{11}(\tilde{p}) + [1 - G_{11}(\tilde{p})] = 0 \\
&\Rightarrow v_1 = \tilde{p} - \frac{1 - G_{11}(\tilde{p})}{g_{11}(\tilde{p})} \qquad (8)
\end{aligned}$$

In equilibrium, buyer 1 should follow the seller's recommendation and offer $\tilde{p} = p_{11}^*$. This means the seller can only choose the way to pool together the information about buyer 2's valuation by pooling recommendations, and the exact recommendations will be determined by the incentive compatible constraints in the resale market. For example, suppose that the seller wants to always allocate the object to buyer 1 and also wants to pool buyer 2's valuation completely. In this case, $G_{11}(v_2) = F(v_2)$. From (8), we obtain $v_1 = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})}$. Denote its solution as p^*_{11}. Then the seller's recommendation for buyer 1 should be $p_{11}(v_2) \equiv p^*_{11}$.

Case 12: In this case, buyer 1 wins and buyer 2 is picked to make the offer. Buyer 2 believes that buyer 1 follows the seller's recommendation regarding the acceptance decision (cf. Equation (3)). The information buyer 2 has when determining the offering price is that he lost and he received recommendations $p_{12}(v_2) = p^*_{12}$ and $A_{11}(p; v_2) = A^*_{11}(p)$. Since v_1 is common knowledge, buyer 2 optimally offers

$$p_{12}(v_2) = \begin{cases} v_1, & if\ v_2 \geq v_1; \\ any\ price\ lower\ than\ v_2, & if\ v_2 < v_1. \end{cases} . \tag{9}$$

This is summarized in the following lemma.

Lemma 3 *In **Case 12**, the price offer is incentive compatible if and only if it satisfies Equation (9); buyer 2 buys the object from buyer 1 at price v_1 if $v_2 \geq v_1$, and buyer 2 does not buy it if $v_2 < v_1$.*

Case 22: In this case, buyer 2 wins and he is picked to make the offer. Buyer 2 believes that buyer 1 will follow the seller's recommendation regarding the acceptance decision (cf. Equation (4)). The information buyer 2 has when deciding the offering price is that he won and he received recommendations $p_{22}(v_2) = p^*_{22}$ and $A_{21}(p; v_2) = A^*_{21}(p)$. Again, since v_1 is common knowledge, buyer 2 optimally offers

$$p_{22}(v_2) = \begin{cases} v_1, & if\ v_2 \leq v_1; \\ any\ price\ higher\ than\ v_2, & if\ v_2 > v_1. \end{cases} \tag{10}$$

This is summarized in the following lemma.

Lemma 4 *In **Case 22**, the price offer is incentive compatible if and only if it satisfies Equation (10); buyer 2 sells the object to buyer 1 at price v_1 if $v_2 \leq v_1$, and buyer 2 keeps the object if $v_2 > v_1$.*

Case 21: In this case, buyer 2 wins and buyer 1 is picked to make the offer. Buyer 1 believes that buyer 2 reports his valuation truthfully, i.e., $\tilde{v}_2 = v_2$, and buyer 2 follows the seller's recommendation regarding the acceptance decision (cf. Equation (5)). The information buyer 1 has when determining the offering price is that he lost and he received recommendations $p_{21}(v_2) = p^*_{21}$ and $A_{22}(p; v_2) = A^*_{22}(p)$. Therefore, he chooses $\tilde{p}$ optimally to maximize:

$$\begin{aligned} & \max_{\tilde{p}} \quad (v_1 - \tilde{p})Prob\{v_2 \leq \tilde{p} | buyer\ 2\ wins,\ p_{21}(v_2) = p^*_{21},\ A_{22}(p; v_2) = A^*_{22}(p)\} \\ \Rightarrow & \max_{\tilde{p}} \quad (v_1 - \tilde{p})Prob\{v_2 \leq \tilde{p} | buyer\ 2\ wins,\ p_{21}(v_2) = p^*_{21}\} \end{aligned} \tag{11}$$

The equality follows from the fact that $A_{22}(v_2; p_{22})$ does not depend on v_2. Let

$$G_{21}(v_2) = F(v_2 | buyer\ 2\ wins, p_{21}(v_2) = p_{21}^*),$$

and

$$g_{21}(v_2) = f(v_2 | buyer\ 2\ wins, p_{21}(v_2) = p_{21}^*).$$

Then the above maximization problem is equivalent to

$$\max_{\tilde{p}} \quad \Pi_{21} = (v_1 - \tilde{p}) G_{21}(\tilde{p}) \tag{12}$$

The incentive compatibility constraint then implies that it is optimal for buyer 1 to follow the seller's recommendation. This is summarized in the following lemma.

Lemma 5 *In **Case 21**, the price offer is incentive compatible if and only if*

$$p_{21}(v_2) = argmax_{\tilde{p}}\{(v_1 - \tilde{p}) G_{21}(\tilde{p})\}, \ \forall v_2;$$

resale occurs at price $p_{21}(v_2)$ if $p_{21}(v_2) > v_2$, and does not occur if $p_{21}(v_2) \leq v_2$.

Note that $p_{21}(v_2)$ is always no greater than buyer 1's valuation v_1. This is because a person will never offer a price higher than his valuation to buy an object. If the first order approach is valid for the maximization problem in (12), its FOC gives us:

$$\begin{aligned} \frac{d\Pi_{21}}{d\tilde{p}} &= -G_{21}(\tilde{p}) + (v_1 - \tilde{p}) g_{21}(\tilde{p}) = 0 \\ &\Rightarrow \quad v_1 = \tilde{p} + \frac{G_{21}(\tilde{p})}{g_{21}(\tilde{p})} \end{aligned} \tag{13}$$

From here, we can solve for the optimal $\tilde{p}$ as a function of the recommendation p_{21}^*. The incentive compatibility constraint implies that buyer 1 should set $\tilde{p} = p_{21}^*$, which then determines p_{21}^*.

3.2 The Off-Equilibrium-Path Continuation Game

In order to determine the incentive compatibility constraints in the initial market, we need to examine the situations when buyer 2 reports his valuation to be $\tilde{v}_2 \neq v_2$. We will focus on buyers' strategies in the resale market. Again, there are four cases.

Case 11: The seller recommends buyer 1 to offer price $p_{11}(\tilde{v}_2)$ and recommends buyer 2 to make acceptance decision according to

$$A_{11}(p; \tilde{v}_2) = \begin{cases} Accept, & if\ p \leq \tilde{v}_2; \\ Reject, & if\ p > \tilde{v}_2. \end{cases}$$

Buyer 1 believes that he is on the equilibrium path and will follow the seller's recommendation to offer $p_{11}(\tilde{v}_2)$. Buyer 2 knows that buyer 1 will offer $p_{11}(\tilde{v}_2)$, and his best response is

$$A_{11}(p; v_2, \tilde{v}_2) = \begin{cases} Accept, & if\ p \leq v_2; \\ Reject, & if\ p > v_2. \end{cases} \tag{14}$$

This is summarized in the following lemma.

Lemma 6 *In **Case 11**, in the off-equilibrium-path continuation game, buyer 1 offers price $p_{11}(\tilde{v}_2)$, and buyer 2 makes his acceptance decision according to Equation (14); resale occurs at price $p_{11}(\tilde{v}_2)$ if $p_{11}(\tilde{v}_2) \leq v_2$ and does not occur if $p_{11}(\tilde{v}_2) > v_2$.*

Case 12: The seller recommends buyer 2 to offer price $p_{12}(\tilde{v}_2)$ and recommends buyer 1 to make his acceptance decision according to

$$A_{12}(p; \tilde{v}_2) = \begin{cases} Accept, & if\ p \geq v_1; \\ Reject, & if\ p < v_1, \end{cases} \tag{15}$$

which is the same as in the analysis in the previous subsection. Buyer 1 believes that he is on the equilibrium path and will follow the seller's recommendation to make acceptance decision according to Equation (15). Buyer 2 believes that buyer 1 will follow recommendation (15), and his optimal price offer is

$$p_{12}(v_2, \tilde{v}_2) = \begin{cases} v_1, & if\ v_2 \geq v_1; \\ any\ price\ lower\ than\ v_1, & if\ v_2 < v_1. \end{cases} \tag{16}$$

This is summarized in the following lemma.

Lemma 7 *In **Case 12**, in the off-equilibrium-path continuation game, buyer 2 offers prices according to Equation (16), and buyer 1 makes acceptance decision according to Equation (15). The induced outcome is that buyer 1 sells the object to buyer 2 at price v_1 when $v_2 \geq v_1$, and buyer 1 keeps the object when $v_2 < v_1$.*

Case 22: The seller recommends buyer 2 to offer price $p_{22}(\tilde{v}_2)$ and recommends buyer 1 to make acceptance decision according to

$$A_{22}(p; \tilde{v}_2) = \begin{cases} Accept, & if\ p \leq v_1; \\ Reject, & if\ p > v_1. \end{cases} \tag{17}$$

Buyer 1 believes that he is on the equilibrium path and will follow the seller's recommendation to make acceptance decision according to Equation (17). Buyer 2 believes that buyer 1 will follow the recommendation (17), and his optimal price offer is

$$p_{22}(v_2) = \begin{cases} v_1, & if\ v_2 \leq v_1; \\ any\ price\ higher\ than\ v_1, & if\ v_2 > v_1. \end{cases} \tag{18}$$

This is summarized in the following lemma.

Lemma 8 *In **Case 22**, in the off-equilibrium-path continuation game, buyer 2 offers prices according to Equation (18), and buyer 1 makes acceptance decision according to Equation (17); buyer 2 sells the object to buyer 1 at price v_1 if $v_2 \leq v_1$ and buyer 2 keeps the object if $v_2 > v_1$.*

Case 21: The seller recommends buyer 1 to offer price $p_{21}(\tilde{v}_2)$ and recommends buyer 2 to make acceptance decision according to

$$A_{21}(p;\tilde{v}_2) = \begin{cases} Accept, & if\ p \geq \tilde{v}_2; \\ Reject, & if\ p < \tilde{v}_2. \end{cases} \tag{19}$$

Buyer 1 believes that he is on the equilibrium path and will follow the seller's recommendation to offer $p_{11}(\tilde{v}_2)$. Buyer 2 believes that buyer 1 will follow price $p_{21}(\tilde{v}_2)$, and his optimal acceptance decision is

$$A_{21}(p;v_2,\tilde{v}_2) = \begin{cases} Accept, & if\ p \geq v_2; \\ Reject, & if\ p < v_2. \end{cases}$$

This is summarized in the following lemma.

Lemma 9 *In **Case 21**, in the off-equilibrium-path continuation game, buyer 1 offers prices $p_{21}(\tilde{v}_2)$, and buyer 2 makes acceptance decision according to Equation (19); resale occurs at price $p_{21}(\tilde{v}_2)$ if $p_{21}(\tilde{v}_2) > v_2$ and does not occur if $p_{21}(\tilde{v}_2) \leq v_2$.*

4 The Initial Market: Establishing IC_2^I, PC_1 and PC_2

In the above section, Lemmas 2-5 characterize the recommendations that are incentive compatible in the resale market. In the following calculations, we will plug in those recommendations whenever possible, except $p_{11}(v_2)$ and $p_{21}(v_2)$ (which we have no explicit solutions). Lemmas 6-9 specify the two buyers' strategies when buyer 2 did not report his valuation truthfully.

Since buyer 1 has no private information, he has no incentive compatibility constraint in the initial market. Suppose that buyer 2 truthfully reports his valuation and also follows the recommendation in the resale market. We can calculate buyer 1's total payoff when he always follows the seller's recommendations in the resale market:

$$\begin{aligned} U_1 \quad = \quad & \int_a^b \Big(x_1(v_2)\left\{\lambda_1\left[v_1 I_{\{v_2<p_{11}(v_2)\}} + p_{11}(v_2) I_{\{v_2\geq p_{11}(v_2)\}}\right] + (1-\lambda_1)v_1\right\} \\ & +x_2(v_2)\left\{\lambda_2(v_1-p_{21}(v_2)) I_{\{v_2\leq p_{21}(v_2)\}}\right\} - t_1(v_2)\Big) dF(v_2), \end{aligned} \tag{20}$$

where $I_{\{.\}}$ is the indicator function. This calculation follows directly from the outcomes in the four cases in the on-equilibrium-path continuation games in the resale market as described in Lemmas 2-5. Rewriting Equation (20), we can obtain the following lemma, which will be useful when formulating the seller's revenue.

Lemma 10 *Player 1's expected payment to the seller can be written as*

$$\int_a^b t_1(v_2)dF(v_2) = \int_a^b \Bigg(x_1(v_2)\left\{\lambda_1\left[v_1 I_{\{v_2<p_{11}(v_2)\}} + p_{11}(v_2)I_{\{v_2\geq p_{11}(v_2)\}}\right] + (1-\lambda_1)v_1\right\}$$

$$+x_2(v_2)\left\{\lambda_2(v_1 - p_{21}(v_2))I_{\{v_2\leq p_{21}(v_2)\}}\right\}\Bigg)dF(v_2) + U_1. \quad (21)$$

We normalize his payoff of not participating to zero. Then buyer 1's participating constraint becomes

$$U_1 \geq 0. \qquad (PC_1) \quad (22)$$

Note that (PC_1) should be binding in the optimal mechanism which maximizes the seller's revenue; if $U_1 > 0$, the seller can obtain a higher revenue by increasing $t_1(v_2)$ while keeping other terms unchanged.

Now suppose that buyer 1 always follows the recommendations. The payoff for buyer 2 if he reports $\tilde{v}_2$ as his valuation (and subsequently acts optimally in the resale market, cf. Lemmas 6-9) is given by

$$\begin{aligned} & U_2(v_2, \tilde{v}_2) \\ = \; & x_1(\tilde{v}_2)\left\{\lambda_1\left[v_2 - p_{11}(\tilde{v}_2)\right]I_{\{v_2\geq p_{11}(\tilde{v}_2)\}} + (1-\lambda_1)(v_2 - v_1)I_{\{v_2\geq v_1\}}\right\} \\ & +x_2(\tilde{v}_2)\left\{(1-\lambda_2)\left[v_2 I_{\{v_2>v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right] + \lambda_2\left[p_{21}(\tilde{v}_2)I_{\{v_2\leq p_{21}(\tilde{v}_2)\}} + v_2 I_{\{v_2>p_{21}(\tilde{v}_2)\}}\right]\right\} \\ & -t_2(\tilde{v}_2) \end{aligned} \quad (23)$$

The formula follows directly from the four cases in the off-equilibrium-path continuation games in the resale market described in Lemmas 6-9. The incentive compatibility constraint and the participation constraint for buyer 2 imply that:

$$U_2(v_2, v_2) \geq U_2(v_2, \tilde{v}_2), \quad \forall\, v_2, \tilde{v}_2 \quad (IC_2^I) \quad (24)$$

$$U_2(v_2, v_2) \geq 0, \qquad \forall\, v_2 \qquad (PC_2) \quad (25)$$

In the following analysis, following the conventions of the mechanism design literature, we first replace IC_2^I with the first order condition of maximizing buyer 2's payoff (23), and then prove that the derived optimal mechanism satisfies IC_2^I. We have the following lemma.

Lemma 11 *Buyer 2's incentive compatibility constraint and participation constraint in the initial market are satisfied only if the following conditions hold:*

$$\begin{aligned} & t_2(v_2) \\ = \; & x_1(v_2)\left\{\lambda_1\left[v_2 - p_{11}(v_2)\right]I_{\{v_2\geq p_{11}(v_2)\}} + (1-\lambda_1)(v_2 - v_1)I_{\{v_2\geq v_1\}}\right\} \end{aligned}$$

$$+x_2(v_2)\left\{(1-\lambda_2)\left[v_2 I_{\{v_2>v_1\}}+v_1 I_{\{v_2\leq v_1\}}\right]+\lambda_2\left[p_{21}(v_2)I_{\{v_2\leq p_{21}(v_2)\}}+v_2 I_{\{v_2>p_{21}(v_2)\}}\right]\right\}$$

$$-\int_a^{v_2}\left\{x_1(\xi)\left[\lambda_1 I_{\{\xi\geq p_{11}(\xi)\}}+(1-\lambda_1)I_{\{\xi\geq v_1\}}\right]+x_2(\xi)\left[(1-\lambda_2)I_{\{\xi>v_1\}}+\lambda_2 I_{\{\xi>p_{21}(\xi)\}}\right]\right\}d\xi$$

$$-U_2(a,a), \tag{26}$$

$$U_2(a,a)\geq 0. \tag{27}$$

The incentive compatibility constraints for buyer 2 together with the allocation rules completely pin down buyer 2's expected payment. Note that buyer 2's informational rent (i.e., payoff $U_2(v_2,v_2)$) is increasing in his valuation. Therefore, buyer 2's participation constraints only need to be satisfied for the lowest type; i.e., $U_2(a,a)\geq 0$. In the optimal mechanism, the participation constraint for the lowest type will be binding; i.e., $U_2(a,a)=0$. If $U_2(a,a)>0$, the seller can increase her revenue by decreasing $U_2(a,a)$.

5 The Seller's Optimization Problem

Making use of the results in the above analysis, the seller maximizes the expected monetary transfers from the two players by picking $t_1(v_2)$, $t_2(v_2)$, $x_1(v_2)$, $x_2(v_2)$, $p_{11}(v_2)$, $p_{12}(v_2)$, $p_{22}(v_2)$, and $p_{21}(v_2)$, subject to the feasibility constraints. The above section shows the implications of the feasibility constraints. We know that monetary transfers are determined by the allocation rules and recommendations. By using Equations (21) and (26), we have

$$\begin{aligned}
& R \\
= & \int_a^b t_1(v_2)dF(v_2)+\int_a^b t_2(v_2)dF(v_2) \\
= & \int_a^b \Big(x_1(v_2)\left\{\lambda_1\left[v_1 I_{\{v_2<p_{11}(v_2)\}}+p_{11}(v_2)I_{\{v_2\geq p_{11}(v_2)\}}\right]+(1-\lambda_1)v_1\right\} \\
& +x_2(v_2)\left\{\lambda_2(v_1-p_{21}(v_2))I_{\{v_2\leq p_{21}(v_2)\}}\right\} \\
& +x_1(v_2)\left\{\lambda_1\left[v_2-p_{11}(v_2)\right]I_{\{v_2\geq p_{11}(v_2)\}}+(1-\lambda_1)(v_2-v_1)I_{\{v_2\geq v_1\}}\right\} \\
& +x_2(v_2)\left\{(1-\lambda_2)\left[v_2 I_{\{v_2>v_1\}}+v_1 I_{\{v_2\leq v_1\}}\right]+\lambda_2\left[p_{21}(v_2)I_{\{v_2\leq p_{21}(v_2)\}}+v_2 I_{\{v_2>p_{21}(v_2)\}}\right]\right\} \\
& -\frac{1-F(v_2)}{f(v_2)}\left\{x_1(v_2)\left[\lambda_1 I_{\{v_2\geq p_{11}(v_2)\}}+(1-\lambda_1)I_{\{v_2\geq v_1\}}\right]\right\} \\
& -\frac{1-F(v_2)}{f(v_2)}\left\{x_2(v_2)\left[(1-\lambda_2)I_{\{v_2>v_1\}}+\lambda_2 I_{\{v_2>p_{21}(v_2)\}}\right]\right\}\Big)dF(v_2) \\
& +U_1+U_2(a,a)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \int_a^b x_1(v_2)\Big\{\lambda_1\left[J_2(v_2)I_{\{v_2\geq p_{11}(v_2)\}} + v_1 I_{\{v_2<p_{11}(v_2)\}}\right] \\
& \qquad + (1-\lambda_1)\left[J_2(v_2)I_{\{v_2\geq v_1\}} + v_1 I_{\{v_2<v_1\}}\right]\Big\}dF(v_2) && (28) \\
& + \int_a^b x_2(v_2)\Big\{(1-\lambda_2)\left[J_2(v_2)I_{\{v_2>v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right] \\
& \qquad + \lambda_2\left[J_2(v_2)I_{\{v_2>p_{21}(v_2)\}} + v_1 I_{\{v_2\leq p_{21}(v_2)\}}\right]\Big\}dF(v_2) \\
& + U_1 + U_2(a,a), && (29)
\end{aligned}
$$

The above equation is intuitive. If the seller can prohibit the resale between the buyers, when she allocates the object to a particular buyer, she gets the virtual valuation of that buyer. However, with resale, she gets the virtual valuation of the final owner instead. For example, if the seller allocates the object to buyer 1 (the speculator) in the initial market, then with probability λ_1 buyer 1 proposes. If the proposed offer is higher than the valuation of buyer 2 (the regular buyer), then buyer 1 will be the final winner; otherwise, buyer 2 will be the final winner. These terms are captured in (28) of equation (29). Note that buyer 1's virtual valuation is v_1, as he has no private information.

It is generally impossible to explicitly characterize the incentive compatible price offers $p_{11}(v_2)$ and $p_{21}(v_2)$. Therefore, characterizing the optimal mechanism to maximize the seller's revenue is not straight-forward. Our approach in this paper is to find an upper bound for the seller's revenue, and then construct a feasible mechanism generating this upper bound revenue. We then can conclude that this mechanism is optimal. The downside for such an approach is that there may exist other optimal mechanisms which are different from our constructed mechanism. Of course, we can always verify whether any explicitly given mechanism is optimal or not, since we know the maximal revenue the seller can achieve.

To find the upper bound seller revenue and construct the optimal mechanism, we need the following four lemmas. Let v_2^* solves $J_2(v_2^*) = v_1$. This v_2^* is the critical valuation for the regular buyer such that his virtual valuation is exactly equal to the speculator's (virtual) valuation. We have the following lemma.

Lemma 12 $J_2(v_2)I_{\{v_2\geq p_{11}(v_2)\}} + v_1 I_{\{v_2<p_{11}(v_2)\}} \leq J_2(v_2)I_{\{v_2\geq v_2^*\}} + v_1 I_{\{v_2<v_2^*\}}$.

The inequality in the lemma follows directly from the fact that the right-hand side is equal to the maximum of $J_2(v_2)$ and v_1, and the left-hand side is equal to either $J_2(v_2)$ or v_1. This lemma implies that, in **Case 11**, it is always the best for the seller to conceal buyer 2's report by making a fully pooling recommendation $p_{11}(v_2) = v_2^*$, assuming that it satisfies the incentive compatibility constraint. Similarly, we have

Lemma 13 $J_2(v_2)I_{\{v_2>p_{21}(v_2)\}} + v_1 I_{\{v_2\leq p_{21}(v_2)\}} \leq J_2(v_2)I_{\{v_2>v_1\}} + v_1 I_{\{v_2\leq v_1\}}$.

This lemma implies that, in **Case 21**, it is always the best for the seller to conceal buyer 2's report by making a fully pooling recommendation $p_{21}(v_2) = v_1$, assuming that it satisfies the incentive compatibility constraint.

From the above two lemmas and inequalities (22) and (27), we have

$$
\begin{aligned}
(29) \quad \leq \quad & \int_a^b x_1(v_2)\Big\{\lambda_1\left[J_2(v_2)I_{\{v_2\geq v_2^*\}} + v_1 I_{\{v_2<v_2^*\}}\right] \\
& \qquad + (1-\lambda_1)\left[J_2(v_2)I_{\{v_2\geq v_1\}} + v_1 I_{\{v_2<v_1\}}\right]\Big\} \\
& + \int_a^b x_2(v_2)\Big\{(1-\lambda_2)\left[J_2(v_2)I_{\{v_2> v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right] \\
& \qquad + \lambda_2\left[J_2(v_2)I_{\{v_2> v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right]\Big\} \\
= \quad & \int_a^b x_1(v_2)\Big\{\lambda_1\left[J_2(v_2)I_{\{v_2\geq v_2^*\}} + v_1 I_{\{v_2<v_2^*\}}\right] \\
& \qquad + (1-\lambda_1)\left[J_2(v_2)I_{\{v_2\geq v_1\}} + v_1 I_{\{v_2<v_1\}}\right]\Big\} \\
& + \int_a^b x_2(v_2)\Big\{(1-\lambda_1)\left[J_2(v_2)I_{\{v_2> v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right] \\
& \qquad + \lambda_1\left[J_2(v_2)I_{\{v_2> v_1\}} + v_1 I_{\{v_2\leq v_1\}}\right]\Big\}. \qquad (30)
\end{aligned}
$$

Similarly to Lemma 12, we have

Lemma 14 $J_2(v_2)I_{\{v_2\geq v_2^*\}} + v_1 I_{\{v_2<v_2^*\}} \geq J_2(v_2)I_{\{v_2\geq v_1\}} + v_1 I_{\{v_2<v_1\}}$.

From this lemma, we obtain

$$
\begin{aligned}
(30) \quad \leq \quad & \int_a^b [x_1(v_2) + x_2(v_2)] \\
& \underbrace{\Big\{\lambda_1\left[J_2(v_2)I_{\{v_2\geq v_2^*\}} + v_1 I_{\{v_2<v_2^*\}}\right] + (1-\lambda_1)\left[J_2(v_2)I_{\{v_2\geq v_1\}} + v_1 I_{\{v_2<v_1\}}\right]\Big\}}_{H(v_2)} dF(v_2) \\
\leq \quad & \int_a^b \max\{H(v_2), 0\} dF(v_2). \qquad (31)
\end{aligned}
$$

Thus, the seller's revenue is bounded by the right hand side of (31), which is denoted as the upper bound revenue. Since this upper bound revenue depends crucially on the function $H(v_2)$, we now examine its properties. Define $\hat{v}_2$ as the unique solution to $\lambda_1 v_1 + (1-\lambda_1)J_2(\hat{v}_2) = 0$ (if any). We have the following lemma.

Lemma 15 ***Situation 1:*** *$a \geq v_1$ and $\lambda_1 v_1 + (1-\lambda_1)J_2(a) < 0$. In this situation,*

$$H(v_2)\begin{cases} < 0, & if\ a \leq v_2 < \hat{v}_2; \\ \geq 0, & if\ \hat{v}_2 \leq v_2 \leq b. \end{cases}$$

Situation 2: *$a < v_1$ and $\lambda_1 v_1 + (1-\lambda_1)J_2(v_1) < 0$. In this situation,*

$$H(v_2)\begin{cases} \geq 0, & if\ a \leq v_2 \leq v_1\ and\ \hat{v}_2 \leq v_2 \leq b; \\ < 0, & if\ v_1 < v_2 < \hat{v}_2. \end{cases}$$

Situation 3: *All other situations: $H(v_2) \geq 0$.*

In Situation 1, $H(v_2)$ is increasing. It is negative for v_2 lower than $\hat{v}_2$ and positive for v_2 higher than $\hat{v}_2$. In Situation 2, $H(v_2)$ is positive except for an interval in the middle. In Situation 3, $H(v_2)$ is always positive.

If a feasible mechanism can generate the upper bound revenue, it would certainly be an optimal mechanism. It turns out that such a mechanism always exits. This mechanism is formulated in the following theorem.

Theorem 1 *The following mechanism maximizes the seller's revenue:*

(i) Allocation rules: $x_2(v_2) = 0$ and

$$x_1(v_2) = \begin{cases} 0, & if\ H(v_2) < 0; \\ 1, & if\ H(v_2) \geq 0; \end{cases} \tag{32}$$

(ii) Resale market offering price recommendations: $p_{11}(v_2) = v_2^*$, $p_{21}(v_2)$ can be any constant, and $p_{12}(v_2)$, $p_{22}(v_2)$ are given by (9), (10), respectively.

(iii) Resale market acceptance recommendations: $A_{11}(p; v_2)$, $A_{12}(p; v_2)$, $A_{22}(p; v_2)$, and $A_{21}(p; v_2)$ are given by (2), (3), (4), and (5), respectively.

(iv) Transfer payments to the seller:

$$\int_a^b t_1(v_2)dF(v_2) = v_1 Prob\{H(v_2) \geq 0\} + \lambda_1(v_2^* - v_1)[1 - F(v_2^*)], \tag{33}$$

$$t_2(v_2) = \begin{cases} 0, & if\ H(v_2) < 0; \\ (1-\lambda_1)(v_2 - v_1)I_{\{v_2 \geq v_1\}} - (1-\lambda_1)\int_a^{v_2} I_{\{H(\xi)\geq 0, \xi \geq v_1\}} d\xi, & if\ H(v_2) \geq 0; \end{cases} \tag{34}$$

(v) The seller's revenue is defined by the right hand side of (31).

This optimal mechanism has many properties, which are discussed in detail below. First, by examining the allocation rule, we have the following striking result.

Corollary 1 *It is never optimal to allocate the object directly to the regular buyer (buyer 2) in the initial market.*

If the speculator has full bargaining power in the resale market, this result is intuitive; the seller can achieve the Myerson revenue by always allocating the object to the speculator. It is striking that the result remains valid even when the speculator has less than full bargaining power. If we interpret the speculator as a dealer, this result simply says that the seller would not sell the object to anyone but the dealer in the optimal mechanism. This is consistent with our observations. In a used car auction, for example, the seller (wholesaler) usually allows only dealers to bid, and is reluctant to deal with individual buyers.

The intuition for this corollary is clearest in Situation 3 of Lemma 15. For the seller, it is always better to generate a final allocation as close to the Myerson allocation as possible because of the Revenue Equivalence Theorem. We know that in the Myerson allocation, the mechanism favors the speculator (buyer 1) in the sense that the speculator wins too often compared to the efficient mechanism. To see the optimality of always allocating the object to the speculator, first suppose that the seller allocates the object to the regular buyer in the initial market. If the regular buyer is picked to propose the offer, then the final allocation will be efficient. If the speculator is picked to propose the offer, then he will not propose an offer greater than his own valuation. As a result, the final allocation will favor the regular buyer and the expected revenue is even less than the efficient allocation. Therefore, allocating the object to the regular buyer is dominated by the efficient allocation. Now suppose that the seller allocates the object to the speculator. If the regular buyer is picked to propose the offer, then the final allocation will be efficient. If the speculator is picked to propose the offer, then he will not propose an offer less than his own valuation. As a result, the final allocation will favor the speculator. In fact, if no additional information except who gets the object is (inevitably) revealed to the resale market, the final allocation will coincide with the Myerson allocation. Therefore, allocating the object to the speculator dominates the efficient allocation, which in turn dominates allocating the object to the regular buyer. Hence, it is never optimal to allocate the object to the regular buyer in the initial market.

There are other properties of the optimal mechanism. First, note that the right hand side of Equation (33) is the speculator's expected benefit from participation. The seller can guarantee the speculator a benefit of v_1 if she does not retain the object, and the speculator can get the extra benefit of $v_2^* - v_1$ if he happens to be picked to propose in the resale market. Of course, all of his expected benefit will be exploited by the seller as his expected payment is set to be equal to his expected benefit from participation. On the other hand, even though the seller does not allocate the object to the regular buyer in the initial market, she nevertheless demands some payment from the regular buyer if she decides not to retain the object. Even so, the regular buyer enjoys a positive payoff, which is higher than the payoff when resale is prohibited. This suggests that the existence of hidden actions (because the resale market is not controlled by the seller) has two effects. First, for the buyer with private information (i.e., the regular buyer), the hidden action problem adds to the adverse selection problem and worsens the seller's revenue. She has to give the regular buyer more informational rents. Second, for the buyer without private information (i.e., the speculator), the existence of hidden action does not benefit him. Therefore, the existence of hidden actions by its own harms the seller only when it is combined with hidden information. Indeed, if both buyers' valuations are common knowledge, then it does not matter to the seller whether the resale can be prohibited or not. In either situation, the seller's revenue will be the same and equal to the maximal of the two valuations.

Second, the transfer payments do not need to be collected at the end of the resale market in the above optimal mechanism. We can make the speculator's transfer payment always equal to his expected benefits (cf. (33)). Then this payment does not depend on v_2, and therefore, it would not reveal the regular buyer's valuation to the speculator. Hence, this payment can be demanded from the speculator in the initial market. Because the speculator has no private information, the seller can demand the payment from the regular buyer at any time, as long as the exact payment amount is concealed from the speculator.

Finally, when resale is not prohibited, the seller cannot generate more than the Myerson revenue. This is because the seller's revenue depends only on the final allocation of the object, and the seller has full control of the final allocation in the Myerson mechanism. Thus, the Myerson revenue establishes another upper bound for the seller's revenue with resale. However, this upper bound may or may not be achievable. This is in contrast to the upper bound defined by the right hand side of (31), which is always achieved in our optimal mechanism. Meanwhile, we can easily find a lower bound for the seller's revenue with resale. It is bounded below by the revenue of a fully efficient mechanism. The seller can guarantee at least this revenue by implementing the fully efficient allocation in the initial market, since no further trade will occur in the resale market given the allocation.

The following corollaries illustrate some important properties of our optimal mechanism. The immediate corollary is on the seller's revenue when she does not retain the object in the optimal mechanism. Let R_M denote the Myerson revenue and R_E denote the (optimal) fully efficient mechanism revenue where the buyer with the higher valuation always wins the object and both the speculator and the lowest valuation regular buyer get zero payoff. We have

Corollary 2 *In* ***Situation 3****, the seller's revenue in the optimal mechanism is* $R = \lambda_1 R_M + (1 - \lambda_1) R_E$*. That is, the seller's maximum revenue is an average of the Myerson revenue and the fully efficient revenue weighted by the speculator's and the regular buyer's bargaining powers* λ_1 *and* $1 - \lambda_1$*.*

This can be seen from (31). In that formula, $J_2(v_2) I_{\{v_2 \geq v_2^*\}} + v_1 I_{\{v_2 < v_2^*\}}$ is the virtual valuation in the Myerson mechanism and $J_2(v_2) I_{\{v_2 \geq v_1\}} + v_1 I_{\{v_2 < v_1\}}$ is the virtual valuation in the optimal efficient mechanism (where the buyer with the higher valuation wins). In the optimal mechanism characterized by Theorem 1 above, the right hand side of (31) is achieved and the two inequalities become binding. In addition, in **Situation 3**, the $H(v_2)$ function is always positive. Therefore, the revenue of this optimal mechanism is the average of the two mechanisms weighted by the two buyers' respective bargaining powers. Here is the intuition. Note that the seller always assigns the object to the speculator in **Situation 3**. If the speculator proposes the offer in resale market (probability

λ_1), then the seller can induce the Myerson's allocation as the final allocation by recommending the speculator to ask a price of $J_2^{-1}(v_1)$, and obtains revenue R_M; if the regular buyer proposes the offer (probability $1-\lambda_1$), he will ask for v_1 for sure, if profitable, as the speculator's valuation is common knowledge, and the final allocation will be efficient providing the seller with revenue R_E.

Note also that $R_M \geq R_E$. Therefore, the seller' revenue is increasing in λ_1 in Situation 3. A similar relationship between the seller's revenue and the speculator's bargaining power holds in all other situations as well, even though a simple explicit formula similarly to the one in the above corollary is not available. We have the following corollary.

Corollary 3 *The seller's revenue is increasing in the speculator's bargaining power λ_1.*

The intuition behind this corollary can be seen by noting that the Myerson allocation generates the highest revenue for the seller. When the speculator has more bargaining power, the final allocation moves closer to the Myerson allocation, and therefore the seller's revenue is higher.

We mentioned earlier that the Myerson revenue may not be achievable in all situations. In fact, the following corollary shows that the Myerson revenue is almost never achievable.

Corollary 4 *The optimal mechanism with resale achieves the Myerson revenue if and only if the speculator has full bargaining power, i.e., $\lambda_1 = 1$.*

The intuition for this result is relatively straight-forward. Obviously, the Myerson revenue cannot be achieved by simply implementing the Myerson allocation in the initial market. This is because buyers will trade further in the resale market, and the trading distorts the final allocation away from the Myerson allocation. However, when the speculator has full bargaining power (in the case he wins, i.e., $\lambda_1 = 1$), the seller can do the following to generate the Myerson revenue. The seller uses the speculator as a middleman by always allocating the object to him, and reveals no information (regarding the regular buyer's valuation) to the resale market. As a result, the speculator's belief about the regular buyer's valuation remains unchanged. Therefore, the speculator offers a price equal to the regular buyer's virtual valuation (evaluated at the speculator's valuation) to the regular buyer in the resale market. In this case, the final allocation coincides with the Myerson allocation. The Revenue Equivalence Theorem implies that this mechanism achieves the Myerson revenue. This means that the Myerson revenue is achievable even if the seller does not have full controlling power over the resale market. This result can be regarded as a special case of Zheng [14].

When $\lambda_1 < 1$, the seller's optimal revenue becomes strictly less than the Myerson revenue. Our Theorem 1 completely characterizes the optimal mechanism in this case, where the Myerson revenue is not attainable. When $\lambda_1 = 0$, surprisingly, the seller can generate no more than the lower bound revenue, i.e., the revenue from the fully efficient mechanism. We have

Corollary 5 *The fully efficient mechanism is optimal if and only if the speculator has no bargaining power (i.e., $\lambda_1 = 0$) in the resale market and $J_2(max\{a, v_1\}) \geq 0$.*

When the condition $J_2(max\{a, v_1\}) \geq 0$ does not hold, the optimal mechanism is a conditional efficient mechanism in the sense that the allocation is efficient between the buyers but the seller may retain the object inefficiently. In the optimal mechanism we constructed in Theorem 1, this revenue is achieved by allocating the object only to the speculator and letting the buyers trade in the resale market. Alternatively, this revenue can be achieved by implementing the efficient allocation directly in the initial market, and no further trade will occur in the resale market. This illustrates that the optimal mechanism we constructed is not the unique optimal mechanism under certain situations.

Now we consider the information revealed to the resale market by the seller through object allocations, transfer payments, and recommendations. As is evidenced in Theorem 1, all recommendations to the speculator do not depend on the regular buyer's reported valuation. We have the following corollary.

Corollary 6 *In the optimal mechanism constructed in Theorem 1, the seller does not reveal any additional information regarding the regular buyer's reported valuation (other than who wins and who loses) to the resale market. Revealing such additional information through transfer payments and recommendations may reduce the seller's revenue.*

If the seller conceals all information regarding the regular buyer's reported valuation, the speculator would set a price equal to the cutoff leading to the Myerson allocation when he is picked to make the offer. Any additional information will prompt the speculator to update his belief, and the price he offers may then roll away from the Myerson cutoff. If the final allocation is different from the Myerson allocation, then the seller cannot obtain the Myerson revenue. If the final allocation coincides with the Myerson allocation, on the other hand, then the seller still obtains the Myerson revenue. For example, in Corollary 5, implementing the fully efficient allocation in the initial market gives the seller the optimal revenue. In this case, even if some private information regarding the regular buyer's valuation is revealed to the resale market, it does not change the final allocation as no resale will occur.

We next consider the possibility of the seller retaining the object. Given the setup of our model, the seller does not retain the object in the Myerson mechanism, because the speculator's virtual value is always positive. However, the seller in the optimal mechanism characterized by Theorem 1 may find it optimal to retain the object under certain circumstances. From Lemma 15, we can determine the conditions for this to happen. We have the following corollary.

Corollary 7 *In **Situation 1**, it is optimal for the seller to retain the object if $a \leq v_2 < \hat{v}_2$. In **Situation 2**, it is optimal for the seller to retain the object if $v_1 \leq v_2 < \hat{v}_2$. In **Situation 3**, it is not optimal for the seller to retain the object. The probability of the seller retaining the object is decreasing in the speculator's bargaining power λ_1.*

The intuition is as follows. We know with resale the seller obtains the virtual valuation of the final owner. The condition in the corollary reflects the situation where the regular buyer would be the final owner and where the regular's virtual valuation is negative if the seller does not retain the object. The seller does not retain the object in the Myerson allocation. In the conditional efficient allocation, however, the seller retains the object in Situations 1 and 3. When λ_1 decreases, the revenue moves towards the conditional efficient allocation, and therefore, the seller retains the object more often.

This corollary partially answers the question of whether allowing resale can improve the overall efficiency of selling. When resale is prohibited, the Myerson allocation is optimal; there is an efficiency loss because the mechanism overly favors the speculator. Meanwhile, the seller does not retain the object inefficiently. When resale is allowed, however, although resale induces a more efficient allocation between the two buyers, there is an efficiency loss from the seller inefficiently retaining the object. Therefore, allowing resale may not necessarily improve the overall efficiency.

Now suppose that the speculator is a pure speculator (with a valuation equal to zero) as in Garratt and Troger [4]. We have the following corollary.

Corollary 8 *If $v_1 = 0$, then the optimal revenue can also be achieved by excluding the speculator in the object allocation.*

The speculator plays an important role in the optimal mechanism, as the seller always allocates the object to him whenever the object is not retained. When the speculator values the object at zero and becomes a pure speculator, however, the seller can also obtain the optimal revenue without the help of the speculator. The seller can simply make an (optimal) take-or-leave-it offer to the regular buyer directly; no resale will occur since the speculator has a valuation of zero. (Note that in Garrett and Troger [4], a pure speculator can still play an active role in standard auctions with resale.) Of course, when the speculator's valuation is non-zero, his role is necessary for the seller to obtain the optimal revenue; excluding him results in a lower seller revenue.

6 Conclusion

In this paper, we construct an optimal mechanism in an environment in which a seller is selling an indivisible object to two buyers. One buyer is a "regular" buyer with a continuous valuation distribution, and the other is a "speculator" or a dealer whose valuation is fixed and known. We focus on the case where the seller cannot prohibit the resale of the object. Following Calzolari and Pavan [2], we model the resale market as an ultimatum bargaining game, with nature picking a proposer randomly to empower each of the buyers with some bargaining strength in the resale market. In this environment, the most striking result is that it is never optimal to assign the object to the regular buyer in the initial market. We also find that the revenue in Myerson's optimal auction can be achieved only when the winner of the initial market has full bargaining power. Furthermore, the seller's revenue is increasing in the winner's bargaining power. Meanwhile, the original seller retains the object more often than in Myerson's optimal auction, as long as the winner does not have full bargaining power, and more and more often when the winner has less and less bargaining power.

In the analysis, we show that the existence of the speculator is very important to the seller. Excluding the speculator from the mechanism usually reduces the seller's revenue unless he is a pure speculator. In the optimal mechanism, the role of the speculator is a middleman. As long as it is in the seller's interest to sell the object, she should always sell it to the speculator.

In this paper, we assume that recommendations can depend on who wins the object in the resale market. We can also assume that the seller cannot make recommendations conditioning on who wins. That means after the initial market concludes, the seller sends $p_{11}(\tilde{v}_2)$, $A_{12}(\tilde{v}_2; p_{12})$, $A_{22}(\tilde{v}_2; p_{22})$ and $p_{21}(\tilde{v}_2)$ to buyer 1; and sends $A_{11}(\tilde{v}_2; p_{11})$, $p_{12}(\tilde{v}_2)$, $p_{22}(\tilde{v}_2)$ and $A_{12}(\tilde{v}_2; p_{21})$ to buyer 2. Note that under this assumption, the seller cannot generate more revenue than the current setting, simply because she has less freedom in the feasible mechanism. However, as we show in this paper, the recommendations in the optimal mechanism do not need to condition on who wins. That means the optimal mechanism we characterize in this paper is optimal in either setup.

We have only one regular buyer in the model. Generalizing the model to more than one regular buyer is not expected to change the qualitative results in the paper. In this case, we need to carefully model the role of each buyer in the resale market. Suppose that there are N *ex-ante* identical regular buyers whose valuations are drawn independently from distribution $F(\cdot)$. If the roles of these regular buyers are also identical in the resale market, the analysis in our paper will apply in a straight-forward way. For example, we can model the resale market follows. If the speculator obtains the object in the initial market, then with probability λ_S he runs a second price auction with an optimal reserve price; with probability $1 - \lambda_S$, he runs a second price auction with a reserve price equal to his own valuation.[10] If one of the regular buyers wins the object in the initial market, the following stochastic ultimatum game takes place between this regular buyer and the speculator: with probability λ_R the regular buyer makes the take-it-or-leave-it offer; with probability $1 - \lambda_R$, the speculator makes the take-it-or-leave-it offer.[11] Note that when $N = 1$, this setup is identical to the model in our paper. To analyze this multiple regular buyer case, we first consider allocating the object to a regular buyer whose valuation is not the highest among those regular buyers. This allocation rule is dominated by the rule of allocating the object to the regular buyer with the highest valuation (while keeping all other allocation rules unchanged). This is simply because there is more revenue to share in the resale market in this way; from the point view of the seller, it is never optimal to allocate the object to a regular buyer whose valuation is not the highest among the regular buyers. Therefore, we can group all regular buyers as one player with distribution $F(\cdot)^N$. The seller is effectively gaming against two buyers: the speculator and a "new" regular buyer with distribution $F(\cdot)^N$. All of the results in our paper can then be applied directly. The seller will always allocate the object to the speculator and reveal no information to the resale market. If the speculator has full bargaining power, the Myerson's allocation is realized; if the "new" regular buyer has full bargaining power, a "conditional" efficient allocation is realized.

A more difficult extension is to have many buyers who are asymmetric. When each buyer has

[10]The speculator has full bargaining power in the first situation and designs an optimal mechanism to resell the object. The speculator has no bargaining power in the second situation and can only choose the best offer from those regular buyers.

[11]Here, we assume that λ_R is the same for each regular buyer to maintain symmetry.

some bargaining power in the resale market, the analysis in this paper is inadequate, and characterizing the optimal mechanism would be an interesting future research project. Of course, our analysis does provide many insights. Without resale, buyers are ranked by their virtual valuations. In contrast, in our simple model with resale, buyers are ranked by the degree of uncertainty in their valuations, regardless of their bargaining power in the resale market. This suggest that in a general model with resale, both the virtual valuations and the variances of valuations are important factors in determining who wins the object in the initial market.

7 Appendix

Proof for Lemma 11

From the Envelope Theorem, we have

$$\begin{aligned}\frac{dU_2(v_2,v_2)}{dv_2} &= x_1(v_2)\left\{\lambda_1 I_{\{v_2\geq p_{11}(v_2)\}} + (1-\lambda_1)I_{\{v_2\geq v_1\}}\right\} \\ &\quad + x_2(v_2)\left\{\lambda_1\left[I_{\{v_2>v_1\}}\right] + (1-\lambda_1)\left[I_{\{v_2>p_{21}(v_2)\}}\right]\right\}. \qquad (35)\end{aligned}$$

Solving the above differential equation gives us

$$\begin{aligned}& U_2(v_2,v_2) \\ = & \int_a^{v_2}\left\{x_1(\xi)\left[\lambda_1 I_{\{\xi\geq p_{11}(\xi)\}} + (1-\lambda_1)I_{\{\xi\geq v_1\}}\right] + x_2(\xi)\left[\lambda_1 I_{\{\xi>v_1\}} + (1-\lambda_1)I_{\{\xi>p_{21}(\xi)\}}\right]\right\}d\xi \\ & + U_2(a,a). \qquad (36)\end{aligned}$$

Substituting Equation (36) into Equation (23) and setting $\tilde{v}_2 = v_2$ yields the desired result. **Q.E.D.**

Proof for Lemma 12

When $v_2 \geq v_2^*$, we have $J_2(v_2) \geq v_1$ since the virtual valuation $J_2(\cdot)$ is increasing. Therefore,

$$\begin{aligned} LHS &\leq J_2(v_2)I_{\{v_2\geq p_{11}(v_2)\}} + J_2(v_2)I_{\{v_2<p_{11}(v_2)\}} \\ &= J_2(v_2)[I_{\{v_2\geq p_{11}(v_2)\}} + I_{\{v_2<p_{11}(v_2)\}}] \\ &= J_2(v_2) = RHS \qquad (37)\end{aligned}$$

When $v_2 < v_2^*$, we have $J_2(v_2) \leq v_1$ since the virtual valuation $J_2(\cdot)$ is increasing. Therefore,

$$\begin{aligned} LHS &\leq v_1 I_{\{v_2\geq p_{11}(v_2)\}} + v_1 I_{\{v_2<p_{11}(v_2)\}} \\ &= v_1[I_{\{v_2\geq p_{11}(v_2)\}} + I_{\{v_2<p_{11}(v_2)\}}] \\ &= v_1 = RHS. \qquad (38)\end{aligned}$$

Q.E.D.

Proof for Lemma 13

Note that when buyer 2 wins the object and buyer 1 makes the offer, buyer 1 will not offer a price higher than his own valuation v_1. Therefore, $p_{21}(v_2) \le v_1$.

When $v_2 \ge v_1$, we have $v_2 \ge p_{21}(v_2)$, and therefore,

$$LHS = J_2(v_2),$$

$$RHS = J_2(v_2).$$

When $v_2 < v_1$, we have $J_2(v_2) < v_1$, and therefore,

$$LHS \le v_1 I_{\{v_2 > p_{21}(v_2)\}} + v_1 I_{\{v_2 \le p_{21}(v_2)\}} = v_1,$$

$$RHS = v_1.$$

Q.E.D.

Proof for Lemma 14

When $v_2 \ge v_2^*$, we have $J_2(v_2) \ge v_1$ since the virtual valuation $J_2(\cdot)$ is increasing. Therefore,

$$LHS = J_2(v_2),$$

and

$$RHS \le J_2(v_2) I_{\{v_2 \ge v_1\}} + J_2(v_2) I_{\{v_2 < v_1\}} = J_2(v_2) = LHS.$$

When $v_2 < v_2^*$, we have $J_2(v_2) \le v_1$ since the virtual valuation $J_2(\cdot)$ is increasing. Therefore,

$$LHS = v_1,$$

$$RHS \le v_1 I_{\{v_2 \ge v_1\}} + v_1 I_{\{v_2 < v_1\}} = v_1 = LHS.$$

Q.E.D.

Proof for Lemma 15

Note that $v_2^* \ge v_1$. When $v_2 \le v_1$, $H(v_2) = \lambda_1 v_1 + (1 - \lambda_1) v_1 \ge 0$.

When $v_2 \ge v_2^*$, $H(v_2) = \lambda_1 J_2(v_2) + (1 - \lambda_1) J_2(v_2) \ge J_2(v_2^*) = v_1 \ge 0$.

When $v_1 < v_2 < v_2^*$, $H(v_2) = \lambda_1 v_1 + (1 - \lambda_1) J_2(v_2)$. In this case, $H(v_2)$ is increasing in v_2 since $J_2(v_2)$ is increasing. The upper bound of $H(v_2)$ is $H(v_2^*) = v_1 \ge 0$. Thus, if $a \ge v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J_2(a) < 0$, then there exists a unique $\hat{v}_2 \in (v_1, v_2^*)$, such that $H(\hat{v}_2) = 0$. If $a < v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J_2(v_1) < 0$, then there exists a unique $\hat{v}_2 \in (a, v_2^*)$, such that $H(\hat{v}_2) = 0$. In all other cases, $a \ge v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J_2(a) \ge 0$ or $a < v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J_2(v_1) \ge 0$, implying $H(v_2) \ge 0$. Summarizing these cases gives us the lemma. **Q.E.D.**

Proof for Theorem 1

Substitute all stated functions into the seller's revenue function (29) and we can obtain the upper bound revenue of the mechanism. Given the stated allocation rules and recommendations, the monetary transfers become

$$\begin{aligned}
\int_a^b t_1(v_2)dF(v_2) &= \int_a^b I_{\{H(v_2)\geq 0\}}\left\{\lambda_1\left[v_1 I_{\{v_2<v_2^*\}}+v_2^* I_{\{v_2\geq v_2^*\}}\right]+(1-\lambda_1)v_1\right\}dF(v_2)\\
&= \int_a^b I_{\{H(v_2)\geq 0\}}\left\{\lambda_1\left[v_1 - v_1 I_{\{v_2\geq v_2^*\}}+v_2^* I_{\{v_2\geq v_2^*\}}\right]+(1-\lambda_1)v_1\right\}dF(v_2)\\
&= \int_a^b I_{\{H(v_2)\geq 0\}}\left\{\lambda_1(v_2^*-v_1)I_{\{v_2\geq v_2^*\}}+v_1\right\}dF(v_2)\\
&= \int_a^b v_1 I_{\{H(v_2)\geq 0\}}dF(v_2)+\int_a^b \lambda_1(v_2^*-v_1)I_{\{v_2\geq v_2^*,H(v_2)\geq 0\}}dF(v_2)\\
&= \int_a^b v_1 I_{\{H(v_2)\geq 0\}}dF(v_2)+\int_a^b \lambda_1(v_2^*-v_1)I_{\{v_2\geq v_2^*\}}dF(v_2)\\
&= v_1 Prob\{H(v_2)\geq 0\}+\lambda_1(v_2^*-v_1)H(v_2^*). \qquad (39)
\end{aligned}$$

If $H(v_2)<0$, then $t_2(v_2)=0$. If $H(v_2)\geq 0$, then

$$\begin{aligned}
t_2(v_2) &= \lambda_1(v_2-v_2^*)I_{\{v_2\geq v_2^*\}}+(1-\lambda_1)(v_2-v_1)I_{\{v_2>v_1\}}\\
&\quad -\int_a^{v_2} I_{\{H(\xi)\geq 0\}}\left[\lambda_1 I_{\{\xi\geq v_2^*\}}+(1-\lambda_1)I_{\{\xi>v_1\}}\right]d\xi\\
&= \lambda_1(v_2-v_2^*)I_{\{v_2\geq v_2^*\}}+(1-\lambda_1)(v_2-v_1)I_{\{v_2>v_1\}}\\
&\quad -\int_a^{v_2} \lambda_1 I_{\{H(\xi)\geq 0,\xi\geq v_2^*\}}d\xi-\int_a^{v_2}(1-\lambda_1)I_{\{H(\xi)\geq 0,\xi>v_1\}}d\xi\\
&= \lambda_1(v_2-v_2^*)I_{\{v_2\geq v_2^*\}}+(1-\lambda_1)(v_2-v_1)I_{\{v_2>v_1\}}\\
&\quad -\int_a^{v_2} \lambda_1 I_{\{\xi\geq v_2^*\}}d\xi-\int_a^{v_2}(1-\lambda_1)I_{\{H(\xi)\geq 0,\xi>v_1\}}d\xi\\
&= \lambda_1(v_2-v_2^*)I_{\{v_2\geq v_2^*\}}+(1-\lambda_1)(v_2-v_1)I_{\{v_2>v_1\}}\\
&\quad -\lambda_1(v_2-v_2^*)I_{\{v_2\geq v_2^*\}}-\int_a^{v_2}(1-\lambda_1)I_{\{H(\xi)\geq 0,\xi>v_1\}}d\xi\\
&= (1-\lambda_1)(v_2-v_1)I_{\{v_2>v_1\}}-(1-\lambda_1)\int_a^{v_2} I_{\{H(\xi)\geq 0,\xi>v_1\}}d\xi. \qquad (40)
\end{aligned}$$

It can be easily verified that the monetary transfers can be translated as follows for each of the three situations.

In **Situation 1**,

$$\int_a^b t_1(v_2)dF(v_2) = v_1[1 - F(\hat{v}_2)] + \lambda_1[1 - F(v_2^*)](v_2^* - v_1),$$

$$t_2(v_2) = \begin{cases} 0, & if\ a \leq v_2 < \hat{v}_2; \\ (1-\lambda_1)(\hat{v}_2 - v_1), & if\ \hat{v}_2 \leq v_2 \leq b; \end{cases}$$

in **Situation 2**,

$$\int_a^b t_1(v_2)dF(v_2) = v_1[1 + F(v_1) - F(\hat{v}_2)] + \lambda_1[1 - F(v_2^*)](v_2^* - v_1),$$

$$t_2(v_2) = \begin{cases} 0, & if\ a \leq v_2 < \hat{v}_2; \\ (1-\lambda_1)(\hat{v}_2 - v_1), & if\ \hat{v}_2 \leq v_2 \leq b; \end{cases}$$

in **Situation 3**,

$$\int_a^b t_1(v_2)dF(v_2) = v_1 + \lambda_1[1 - F(v_2^*)](v_2^* - v_1),$$

$$t_2(v_2) = \begin{cases} (1-\lambda_1)(a - v_1), & if\ a \geq v_1; \\ 0, & if\ a < v_1. \end{cases}$$

The only thing left to show is that this mechanism satisfies all of the incentive compatibility constraints and the participation constraints.

For the resale market, since $A_{11}(p; v_2)$, $A_{12}(p; v_2)$, $A_{22}(p; v_2)$, $A_{21}(p; v_2)$, $p_{12}(v_2)$, $p_{22}(v_2)$ are directly taken from the lemmas on the buyers' incentive compatible constraints, we only need to check that $p_{11}(v_2)$ and $p_{21}(v_2)$ are incentive compatible. To proceed, we will examine each of the three situations separately.

In **Situation** 1, we first examine $p_{11}(v_2)$. The winner, if any, is always buyer 1 in the initial market and the recommendations are fully pooling. Thus, when buyer 1 chooses the price to offer in Case 11, he believes that buyer 2's valuation is less than $\hat{v}_2$, i.e., $G_{11}(v_2) = \frac{F(v_2) - F(\hat{v}_2)}{1 - F(\hat{v}_2)}$. Thus, substituting $G_{11}(v_2) = F(v_2)$ into the FOC (8) determines the price offer, i.e., $v_1 = \tilde{p} - \frac{1 - \frac{F(\tilde{p}) - F(\hat{v}_2)}{1 - F(\hat{v}_2)}}{\frac{f(\tilde{p})}{1 - F(\hat{v}_2)}} = J_2(\tilde{p})$. Since we assumed that $J_2(a) \leq v_1 \leq J_2(b)$ and that $J_2(v_2)$ is increasing, there exists a unique solution $\tilde{p} = v_2^*$. It is indeed optimal to for buyer 1 to offer price v_2^*.

We now examine $p_{21}(v_2)$. Since only buyer 1 can win in the initial market, Case 21 is off the equilibrium path, and therefore $p_{21}(v_2)$ is not relevant as long as it does not reveal information.

We need last to verify buyer 2's incentive compatibility constraint in the initial market. Substituting all the relevant functions into Equation (23), we have

$$
\begin{aligned}
& U_2(v_2, \tilde{v}_2) \\
= & \begin{cases} 0, & if \;\; a \le v_2 < \hat{v}_2 \\ \lambda_1 \left[v_2 - v_2^*\right] I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - v_1) - (1-\lambda_1)(\hat{v}_2 - v_1); & if \;\; \hat{v}_2 \le v_2 \le b \end{cases} \\
= & \begin{cases} 0, & if \;\; a \le v_2 < \hat{v}_2; \\ \lambda_1 \left(v_2 - v_2^*\right) I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2), & if \;\; \hat{v}_2 \le v_2 \le b. \end{cases}
\end{aligned} \tag{41}
$$

First consider $v_2 \le \hat{v}_2$. Note that $v_2 \le v_2^*$, since $\hat{v}_2 \le v_2^*$. Truthful reporting by buyer 2 implies that $U_2(v_2, v_2) = 0$. If he deviates to any $\tilde{v}_2 \le \hat{v}_2$, it gives him the same payoff of 0. If he deviates to $\tilde{v}_2 \ge \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda_1)(v_2 - \hat{v}_2) \le 0.$$

Thus, buyer 2 has no incentive to deviate.

Now consider $v_2 \ge \hat{v}_2$. In this case, $v_2 \ge v_1$. Truthful reporting by buyer 2 implies that

$$U_2(v_2, v_2) = \lambda_1 \left(v_2 - v_2^*\right) I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2) \ge 0. \tag{42}$$

If buyer 2 deviates to any $\tilde{v}_2 \ge \hat{v}_2$, then it give him the same payoff. If he deviates to $\tilde{v}_2 \le \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate.

In **Situation** 2, we first examine $p_{11}(v_2)$. The winner, if any, is always buyer 1 in the initial market and the recommendation is fully pooling. Thus, when buyer 1 chooses the price to offer in Case 11, he believes that buyer 2's valuation is always in $[a, v_1] \cup [\hat{v}_2, b]$, i.e.,

$$
G_{21}(v_2) = \begin{cases} \frac{F(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & if \;\; a \le v_2 \le v_1; \\ \\ \frac{F(v_1)}{1-F(\hat{v}_2)+F(v_1)}, & if \;\; v_1 < v_2 < \hat{v}_2; \\ \\ \frac{F(v_2)-F(\hat{v}_2)+F(v_1)}{1-F(\hat{v}_2)+F(v_1)}, & if \;\; \hat{v}_2 \le v_2 \le b; \end{cases}
$$

and

$$
g_{21}(v_2) = \begin{cases} \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & if \;\; a \le v_2 \le v_1; \\ \\ 0, & if \;\; v_1 < v_2 < \hat{v}_2; \\ \\ \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & if \;\; \hat{v}_2 \le v_2 \le v_1; \end{cases}
$$

$$\frac{1-G_{21}(v_2)}{g_{21}(v_2)} = \begin{cases} \frac{1-F(\hat{v}_2)+F(v_1)-F(v_2)}{f(v_2)}, & if\ a \le v_2 \le v_1; \\ \\ +\infty, & if\ v_1 < v_2 < \hat{v}_2; \\ \\ \frac{1-F(v_2)}{f(v_2)}, & if\ \hat{v}_2 \le v_2 \le b. \end{cases}$$

Thus, by replacing these functions, the FOC (8) determining the price offer becomes

$$v_1 = \begin{cases} J_2(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})}, & if\ a \le v_2 \le v_1; \\ \\ -\infty, & if\ v_1 < v_2 < \hat{v}_2; \\ \\ J_2(\tilde{p}), & if\ \hat{v}_2 \le v_2 \le b. \end{cases} \cdot \tag{43}$$

This equation has a unique solution. First, note that $J_2(a) \le v_1 \le J_2(b)$, $J_2(v_2)$ increasing, $J_2(\hat{v}_2) = 0$, and $J_2(v_2^*) = v_1$. If $a \le v_2 \le v_1$, RHS (of (43))$< J_2(\tilde{p}) < J_2(v_2^*) = v_1$, and therefore there is no solution to (43). If $v_1 < v_2 < \hat{v}_2$, obviously there is no solution to (43). If $\hat{v}_2 \le v_2 \le b$, there is a unique solution $\tilde{p} = v_2^*$. We now need to show that $\tilde{p} = v_2^*$ is a global maxima. First,

$$\begin{aligned} \frac{d\Pi_1}{d\tilde{p}} &= v_1 g_{21}(\tilde{p}) - \tilde{p} g(\tilde{p}) + [1 - G_{11}(\tilde{p})] \\ &= g_{21}(\tilde{p}) \left[v_1 - \tilde{p} + \frac{1-G_{21}(\tilde{p})}{g_{21}(\tilde{p})} \right] \\ &= \begin{cases} g_{21}(\tilde{p}) \left[v_1 - J_2(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})} \right], & if\ a \le v_2 \le v_1; \\ \\ 1 - G_{11}(\tilde{p}), & if\ v_1 < v_2 < \hat{v}_2; \\ \\ v_1 - J_2(\tilde{p}), & if\ \hat{v}_2 \le v_2 \le b. \end{cases} \end{aligned} \tag{44}$$

Since $v_1 = J_2(v_2^*)$ and $J_2(\cdot)$ is increasing, $\frac{d\Pi_1}{d\tilde{p}} \ge 0$ if $\tilde{p} \le v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \le 0$ if $\tilde{p} \ge v_2^*$. Thus, it is indeed optimal to follow the recommendation and offer price v_2^*.

We now examine $p_{21}(v_2)$. Since only buyer 1 can win in the initial market, Case 21 is off the equilibrium path.

Finally, we verify buyer 2's incentive compatibility constraint in the initial market. Substituting all relevant functions into Equation (23), we have

$$\begin{aligned} & U_2(v_2, \tilde{v}_2) \\ = & \begin{cases} 0, & if\ a \le v_2 \le v_1 \\ 0, & if\ v_1 \le v_2 < \hat{v}_2 \\ \lambda_1 [v_2 - v_2^*] I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - v_1) - (1-\lambda_1)(\hat{v}_2 - v_1), & if\ \hat{v}_2 \le v_2 \le b \end{cases} \end{aligned}$$

$$= \begin{cases} 0, & if\ a \le v_2 < \hat{v}_2; \\ \lambda_1 (v_2 - v_2^*) I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2), & if\ \hat{v}_2 \le v_2 \le b. \end{cases} \tag{45}$$

First consider $v_2 \le \hat{v}_2$. Note that $v_2 \le v_2^*$, since $\hat{v}_2 \le v_2^*$. Truthful reporting by buyer 2 implies that $U_2(v_2, v_2) = 0$. If buyer 2 deviates to any $\tilde{v}_2 \le \hat{v}_2$, it gives him the same payoff of 0. If he deviate to $\tilde{v}_2 \ge \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda_1)(v_2 - \hat{v}_2) \le 0.$$

Thus, he has no incentive to deviate.

Now consider $v_2 \ge \hat{v}_2$. In this case, $v_2 \ge v_1$. Truthful reporting implies that

$$U_2(v_2, v_2) = \lambda_1 (v_2 - v_2^*) I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2) \ge 0. \tag{46}$$

If he deviates to any $\tilde{v}_2 \ge \hat{v}_2$, he obtains the same payoff. If he deviates to $\tilde{v}_2 \le \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate.

In **Situation** 3, we first examine $p_{11}(v_2)$. Since only buyer 1 can be the winner in the initial market and the recommendation is fully pooling, buyer 1 receives no additional information about buyer 2's valuation, i.e., buyer 1's valuation remains: $G_{11}(v_2) = F(v_2)$. Thus, by substituting $G_{11}(v_2) = F(v_2)$, the FOC (8) determining the price offer becomes $v_1 = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})} = J_2(\tilde{p})$. Since we assumed that $J_2(a) \le v_1 \le J_2(b)$ and that $J_2(v_2)$ increasing, there exists a unique solution $\tilde{p} = v_2^*$. We now show that $\tilde{p} = v_2^*$ is a global maxima. First,

$$\begin{aligned} \frac{d\Pi_1}{d\tilde{p}} &= v_1 f(\tilde{p}) - \tilde{p} f(\tilde{p}) + [1 - F(\tilde{p})] \\ &= f(\tilde{p}) \left[v_1 - \tilde{p} + \frac{1-F(\tilde{p})}{f(\tilde{p})} \right] \\ &= f(\tilde{p}) \left[v_1 - J_2(\tilde{p}) \right]. \end{aligned} \tag{47}$$

Since $v_1 = J_2(v_2^*)$ and $J_2(\cdot)$ increasing, $\frac{d\Pi_1}{d\tilde{p}} \ge 0$ if $\tilde{p} \le v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \le 0$ if $\tilde{p} \ge v_2^*$. Thus, it is indeed optimal for buyer 1 to follow the recommendation and offer price v_2^*.

We now examine $p_{21}(v_2)$. Since only buyer 1 can win in the initial market, Case 21 is off the equilibrium path. The optimality of price offer $p_{21}(v_2)$ can be supported by buyer 1's belief $v_2 = b$.

Finally, we verify buyer 2's incentive compatibility constraint in the initial market. Substituting all relevant functions into Equation (23), we have

$$\begin{aligned} & U_2(v_2, \tilde{v}_2) \\ = & \left\{ \lambda_1 [v_2 - v_2^*] I_{\{v_2 \ge v_2^*\}} + (1-\lambda_1)(v_2 - v_1) I_{\{v_2 \ge v_1\}} \right\} - \max\{(1-\lambda_1)(a - v_1), 0\}. \end{aligned} \tag{48}$$

This payoff function does not depend on $\tilde{v}_2$, and therefore, buyer 2 has no incentive to lie about his valuation. **Q.E.D.**

Proof for Corollary 3
For **Situation 1**, the seller's revenue is

$$\begin{aligned} R &= \int_{\hat{v}_2}^{v_2^*} [\lambda_1 v_1 + (1-\lambda_1) J_2(v_2)]\, dF(v_2) + \int_{v_2^*}^{b} [\lambda_1 J_2(v_2) + (1-\lambda_1) J_2(v_2)]\, dF(v_2) \\ &= \int_{\hat{v}_2}^{v_2^*} [\lambda_1 v_1 + (1-\lambda_1) J_2(v_2)]\, dF(v_2) + \int_{v_2^*}^{b} J_2(v_2) dF(v_2). \end{aligned} \tag{49}$$

Thus,

$$\begin{aligned} \frac{dR}{d\lambda_1} &= -\frac{d\hat{v}_2}{d\lambda_1} [\lambda_1 v_1 + (1-\lambda_1) J_2(\hat{v}_2)]\, f(v_2) + \int_{\hat{v}_2}^{v_2^*} [v_1 - J_2(v_2)]\, dF(v_2) \\ &= \int_{\hat{v}_2}^{v_2^*} [v_1 - J_2(v_2)]\, dF(v_2) \geq 0. \end{aligned} \tag{50}$$

The last equality above follows from the definition of $\hat{v}_2$ and the inequality follows from the assumption that $J_2(v_2)$ is increasing and that $J_2(v_2^*) = v_1$.

For **Situation 2**,

$$\begin{aligned} R &= \int_{a}^{v_1} [\lambda_1 v_1 + (1-\lambda_1) v_1]\, dF(v_2) \\ &\quad + \int_{\hat{v}_2}^{v_2^*} [\lambda_1 v_1 + (1-\lambda_1) J_2(v_2)]\, dF(v_2) \\ &\quad + \int_{v_2^*}^{b} [\lambda_1 J_2(v_2) + (1-\lambda_1) J_2(v_2)]\, dF(v_2) \\ &= v_1 F(v_1) + \int_{\hat{v}_2}^{v_2^*} [\lambda_1 v_1 + (1-\lambda_1) J_2(v_2)]\, dF(v_2) + \int_{v_2^*}^{b} J_2(v_2) dF(v_2). \end{aligned} \tag{51}$$

Thus,

$$\begin{aligned} \frac{dR}{d\lambda_1} &= -\frac{d\hat{v}_2}{d\lambda_1} [\lambda_1 v_1 + (1-\lambda_1) J_2(\hat{v}_2)]\, f(v_2) + \int_{\hat{v}_2}^{v_2^*} [v_1 - J_2(v_2)]\, dF(v_2) \\ &= \int_{\hat{v}_2}^{v_2^*} [v_1 - J_2(v_2)]\, dF(v_2) \geq 0. \end{aligned} \tag{52}$$

For **Situation 3**,

$$R = \int_a^{v_2^*} [\lambda_1 v_1 + (1-\lambda_1) J_2(v_2)]\, dF(v_2) + \int_{v_2^*}^{b} [\lambda_1 J_2(v_2) + (1-\lambda_1) J_2(v_2)]\, dF(v_2). \tag{53}$$

Thus,

$$\frac{dR}{d\lambda_1} = \int_a^{v_2^*} [v_1 - J_2(v_2)]\, dF(v_2) \geq 0.$$

To summarize, since the revenue is a continuous function of λ_1, and in each case it is increasing in λ_1, it must be increasing in λ_1 over its entire domain. **Q.E.D.**

Proof for Corollary 7

Note that $\hat{v}_2$ is determined by $\lambda_1 v_1 + (1-\lambda_1) J_2(\hat{v}_2) = 0$. The immediate implication is that $J_2(\hat{v}_2) < 0$. Differentiating both sides with respect to λ_1 yields:

$$v_1 - J_2(\hat{v}_2) + (1-\lambda_1) J_2{}'(\hat{v}_2) \frac{\partial \hat{v}_2}{\partial \lambda_1} = 0 \Leftrightarrow \frac{\partial \hat{v}_2}{\partial \lambda_1} = \frac{J_2(\hat{v}_2) - v_1}{(1-\lambda_1) J_2{}'(\hat{v}_2)} < 0$$

Since the probability for the seller to retain the object is nondecreasing in $\hat{v}_2$ in all three situations, that probability is also decreasing in λ_1. **Q.E.D.**

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