Rationalizability in General Situations*

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Abstract

We present an analytical framework that can be used to study rationalizable strategic behavior in arbitrary strategic games with various modes of behavior. We show that, under mild conditions, the notion of rationalizability defined in general situations has nice properties similar to those in finite games. The major features of this paper are (i) our approach does not require any kind of technical assumptions on the structure of the game, and (ii) the analytical framework provides a unified treatment of players' general preferences, including expected utility as a special case. In this paper, we also investigate the relationship between rationalizability and Nash equilibrium in general games.

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1 Introduction

The notion of rationalizability proposed by Bernheim (1984) and Pearce (1984) is one of the most important and fundamental solution concepts in non-cooperative game theory. The basic idea behind this notion is that rational behavior must be justified by "rational beliefs" and conversely, "rational beliefs" must be based on rational behavior. The notion of rationalizability captures the strategic implications of the assumption of "common knowledge of rationality" (see Tan and Werlang (1988)), which is different from the assumption of "commonality of beliefs" or "correct conjectures" in an equilibrium (see Aumann and Brandenburger (1995)).

In the literature, most of the studies of rationalizable strategic behavior have been restricted to finite games.[1] The main purpose of this paper is to extend the notion of rationalizability to general games that inherits the properties of conventional rationalizability. Since many important models arising in economic applications are games with infinite strategy spaces and discontinuous payoff functions, e.g., models of price and spatial competition, auctions, and mechanism design,[2] it is clearly important and practically relevant to extend the notion of rationalizability to arbitrary games.

In the definition of conventional rationalizability, each player is implicitly assumed to be Bayesian rational – i.e., each player maximizes the expected utility given his probabilistic belief about the opponents' strategy choices. The Ellsberg Paradox and related experimental evidence demonstrate that a decision maker usually displays an aversion to uncertainty or ambiguity and, thereby, motivates gen-

[1] Berheim (1984, Proposition 3.2) and Tan and Werlang (1988) studied the properties of rationalizable strategies in compact (metric) and continuous games. There are also a few exceptional examples on infinite games such as Arieli's (2010) analysis of rationalizability in continuous games where every player's strategy set is a Polish space and the payoff function of each player is bounded and continuous and Zimper's (2006) discussions on a variant of "strong point-rationalizability" in games with metrizable strategy sets. See also Jara-Moroni (2012) and Yu (2010) for discussions on rationalizability in games with a continuum of players.

[2] See, e.g., Bergemann and Morris (2005a, 2005b), Bergemann et al. (2011), and Kunimoto and Serrano (2011). In particular, Bergemann et al. (2011) and Kunimoto and Serrano (2011) considered infinite mechanisms (game forms) for which transfinite rounds of deletion of never-best replies or dominated strategies are necessary.

eralizations of the subjective expected utility model. Epstein (1997) extended the concept of rationalizability to a variety of general preference models by characterizing rationalizability and survival of iterated deletion of never best response strategies as the (equivalent) implications of rationality and common knowledge of rationality. In his analysis Epstein offered a "model of preference" to allow for different categories of "regular" preferences such as subjective expected utility, probabilistically sophisticated preference, Choquet expected utility and the multi-priors model. However, from a technical point of view, Epstein's (1997) analysis relies on topological assumptions on the game structure and, in particular, his discussions on rationalizability are restricted to finite games.[3] Apt (2007) relaxed the finite set-up of games and studied rationalizability by an iterative procedure, but his analysis implicitly requires players' preferences to have a certain form of expected utility. In this paper we study rationalizable strategic behavior in general situations: arbitrary games with various modes of behavior.

To define the notion of rationalizability, we need to consider a system of preferences/beliefs for possible subgame situations. By using Harsanyi's (1967-68) notion of type, we introduce the simple analytical framework – the "model of type," which specifies a set of admissible and feasible types for each of players in every possible restriction of game situation. For each type of a player, the player is able to make a choice decision over his own strategies. Our approach is topology-free and is applicable to arbitrary games with various modes of behavior.

In a related paper, Apt and Zvesper (2010) provided a broad and general approach to various forms of customary iterative solution concepts in arbitrary strategic games with a special emphasis on the role of monotonicity in "rationality." Our analysis of rationalizable strategic behavior is, in this respective, harmonious with Apt and Zvesper's (2010) approach. As we have emphasized, this paper focuses on how to extend the notion of rationalizability to general games with various modes of behavior that inherits the properties of conventional rationalizability, while Apt

[3] See also Asheim's (2006) related discussions on rationalizability under alternative epistemological assumptions.

and Zvesper's (2010) paper focuses on examining and comparing, in the context of epistemic analysis with possibility correspondences, various forms of customary iterative solution concepts in arbitrary strategic games through the monotonic property of "rationality" behind the iterated dominance notions.

We offer a definition of rationalizability in general situations (Definition 1). Roughly speaking, a set of strategy profiles is regarded as "rationalizable" if every player's strategy in this set can be justified by some of the player's types associated with the set. We show that the union of all the rationalizable sets is the largest (w.r.t. set inclusion) rationalizable set in the product form (Proposition 1), which can be derived from an iterated elimination of never-best responses (IENBR). Moreover, IENBR is an order-independent procedure (Proposition 2). In addition, we study the epistemic foundation of rationalizability in general situations: We formulate and prove that rationalizability is the strategic implication of common knowledge of rationality (Proposition 4). We show an equivalence between the notion of rationalizability and the notion of a posteriori equilibrium in general settings (Proposition 5).

In this paper, we also investigate the relationship between rationalizability and Nash equilibrium. We demonstrate through an example that the IENBR procedure may generate spurious Nash equilibrium and, then, present a necessary and sufficient condition for no spurious Nash equilibria (Proposition 3). In dominance-solvable games, the unique Nash equilibrium can be obtained by IENBR and, moreover, rationalizable strategic behavior in a wide range of preference models is observationally indistinguishable from Nash equilibrium behavior (Proposition 6). We show that, through examples, rationalizability neither implies nor is implied by iterated strict dominance defined by Chen et al. (2007) in general game situations. It is worthwhile to emphasize that one important feature of this paper is that, throughout this paper, we do not require any kind of technical assumptions on the structure of the game or particular strong behavioral assumptions on players' preferences; in particular, we do not require the compactness, convexity, and continuity conditions on strategy sets and payoff functions, and we do not even assume that players' preferences have utility function representations.

The rest of this paper is organized as follows. Section 2 is the set-up. Sections 3 and 4 present the main results concerning rationalizability with IENBR and Nash equilibrium respectively. Section 5 provides the epistemic foundation for rationalizability. Section 6 discusses the relationship between rationalizability and iterated strict dominance. Section 7 offers concluding remarks.

2 Set-up

Consider a normal-form game:

$$G \equiv (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}),$$

where N is an (in)finite set of players, S_i is an (in)finite set of player i's strategies, and $u_i \colon S \to \mathbb{R}$ is player i's arbitrary payoff function where $S \equiv \times_{i \in N} S_i$.[4] For $s \in S$ let $s \equiv (s_i, s_{-i})$. A strategy profile s^* in S is a *(pure) Nash equilibrium* in G if for every player i,

$$u_i(s^*) \geq u_i\left(s_i, s^*_{-i}\right) \ \forall s_i \in S_i.$$

The notion of "type" due to Harsanyi (1967-68) is a simple and parsimonious description of the exhaustive uncertainty facing a player, including the player's knowledge, preferences/beliefs, etc.[5] Given player i's type, the player has one corresponding preference relation over his own strategies, according to which the player can make his choice decision. We consider a *model of type for game G*:

$$\top^G \equiv \{\top_i^G(\cdot)\}_{i \in N},$$

[4]Throughout this paper, we consider only the sets which satisfy the ZFC axioms; see, e.g., Aliprantis and Border (2006).

[5]As Harsanyi (1967-68, p.171) pointed out, "we can regard the [type] as representing certain physical, social, and psychological *attributes* of player *i himself* in that it summarizes some crucial parameters of player i's own payoff function U_i as well as the main parameters of his beliefs about his social and physical environment." Harsanyi demonstrated that games with incomplete information can be solved by using the notion of type. This notion is also useful to analyze strategic uncertainty about the actual play of the games with complete information. In Section 5, we will introduce a more precise and formal notion of "type" in the epistemic analysis of rationalizable strategic behavior.

where $\top_i^G(\cdot)$ is defined for every (nonempty) subset $S' \subseteq S$ and every player $i \in N$. The set $\top_i^G(S')$ is interpreted as player i's *type space* in the reduced game $G|_{S'} \equiv (N, \{S'_i\}_{i\in N}, \{u_i|_{S'}\}_{i\in N})$, where $u_i|_{S'}$ is the payoff function u_i restricted on S'. In other words, $\top_i^G(S')$ is the set of all plausible types of player i when the player faces strategic uncertainty about the other players' actions in $S'_{-i} \equiv \{s_{-i}|\ (s_i, s_{-i}) \in S'\}$.

Each type $t_i \in \top_i^G(S')$ has a corresponding *preference relation (or binary relation)* $\succsim_{t_i}$ over player i's strategies in S_i. The indifference relation, $\sim_{t_i}$, is defined as usual, i.e., $s_i \sim_{t_i} s'_i$ iff $s_i \succsim_{t_i} s'_i$ and $s'_i \succsim_{t_i} s_i$. For instance, we may consider $\top_i^G(S')$ as a probability space or a regular preference space defined on S'. The following example demonstrates that this analytical framework can be applied to finite games where the players have the standard subjective expected utility (SEU) preferences.

Example 1. Consider a finite game G. Player i's belief about the strategies that the opponents play in the reduced game $G|_{S'}$ is defined as a probability distribution μ_i over S'_{-i}, i.e., $\mu_i \in \Delta\left(S'_{-i}\right)$ where $\Delta\left(S'_{-i}\right)$ is the set of probability distributions over S'_{-i}. For any μ_i, the expected payoff of s_i can be calculated by

$$U_i\left(s_i, \mu_i\right) = \sum_{s_{-i}\in S'_{-i}} \mu_i\left(s_{-i}\right) \cdot u_i\left(s_i, s_{-i}\right)$$

where $\mu_i\left(s_{-i}\right)$ is the probability assigned by μ_i to s_{-i}. That is, μ_i generates an SEU preference over S_i. For our purpose we may define a model of type (on G) as follows:

$$\top^G = \{\top_i^G(\cdot)\}_{i\in N},$$

where, for every player $i \in N$, $\top_i^G(S') = \Delta\left(S'_{-i}\right)$ for every (nonempty) subset $S' \subseteq S$. Note that the beliefs are "correlated" in the sense that a belief is represented by a joint probability distribution over the opponents' strategies. The model of type allows to represent player's beliefs as product (independent) or degenerated (point) probability distributions over opponents' strategies.[6]

[6]In the game-theory literature, players are typically assumed to be Bayesian rational; that is, each player forms a prior over the space of states of the world and maximizes the expected value of some fixed vNM index on outcomes. The model of type also allows to represent player's beliefs as other forms of subjective expected utility preferences such as Borgers's (1993) ordinal expected utility and the state-dependent utility preferences discussed in Morris and Takahashi (2011).

For $t_i \in \top_i^G(S')$, a strategy $s_i \in S_i$ is *one of most preferred actions for* t_i if $s_i \succsim_{t_i} s_i'$ for all $s_i' \in S_i$. (Notice that even if a reduced game $G|_{S'}$ is concerned, any strategy of player i in the original game G can be a candidate for the most preferred choices.) Let

$$\beta(t_i) \equiv \{s_i \in S_i |\ s_i \succsim_{t_i} s_i' \text{ for all } s_i' \in S_i\}.$$

We first present a definition of rationalizability in a game G with the type model $\top^G$. The spirit of this definition is that for every strategy in a rationalizable set, the player can always find some type in the type space defined over this set to support his choice of strategy.

Definition 1. A subset $R \subseteq S$ is *rationalizable in* $\top^G$ if $\forall i$ and $\forall s \in R$, there exists some $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$.

For our discussions, we need the following two conditions on the model of type $\top^G$.

C1 (Monotonicity) $\forall i$, $\top_i^G(S') \subseteq \top_i^G(S'')$ if $S' \subseteq S''$.

The monotonicity condition requires that when one player faces a greater degree of strategic uncertainty, the player possesses more types available for resolving uncertainty. In the literature on information economics, a type of a player is interpreted as the initial private information, about all the uncertainty regarding the state of nature in a game situation, that player has. From this point of view, it is natural to assume that there are more types available if there is more strategic uncertainty about the choices of the players. Under C1, $\top_i^* \equiv \top_i^G(S)$ can be viewed as the "universal" type space of player i in game G.

For $s \in S$, player i's *Dirac type* $\delta_i(s)$ is a type with the property:

$$\forall s_i', s_i'' \in S_i, u_i(s_i', s_{-i}) \geq u_i(s_i'', s_{-i}) \text{ iff } s_i' \succsim_{\delta_i(s)} s_i''.$$

A Dirac type $\delta_i(s)$ is a degenerated type with which player i behaves as if he faces a certain play s_{-i} of the opponents; in probabilistic models, a Dirac type is a point mass that represents a point belief about the opponents' choices. Observe that s^* is a Nash equilibrium iff, for every player i, s_i^* is a best response to $\delta_i(s^*)$. The following condition requires that the only possible type for a deterministic case – i.e. a singleton of a certain play of the opponents – is a Dirac type. This condition is a rather natural requirement when strategic uncertainty is reduced to a special deterministic case of certainty.

C2 (Diracability) $\forall i$, $\top_i^G(\{s\}) = \{\delta_i(s)\}$ for all $s \in S$.

In Example 1, it is easy to see that C1 and C2 are satisfied for the standard SEU preference model. C1 and C2 imply that $\delta_i(s) \in \top_i^G(S')$ if $s \in S'$, i.e., the type space on S' contains all the possible Dirac types on S'.

We call a strategy profile rationalizable in $\top^G$ if this profile lies in a rationalizable set in $\top^G$. The following Proposition 1 asserts that, under C1, there is the largest (w.r.t. set inclusion) rationalizable set in product form that, under C2, contains all the Nash equilibria in G.

Proposition 1. *Under C1, $R^* \equiv \cup_{R \text{ is rationalizable in } \top^G} R$ is the largest (product-form) rationalizable set in $\top^G$. Under C2, R^* contains all the Nash equilibria in G.*

Proof: It suffices to show that R^* is a rationalizable set in $\top^G$. Let $s \in R^*$. Then, there exists a rationalizable set R in $\top^G$ such that $s \in R$. Thus, for every player i, there exists some $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$. Since $R \subseteq R^*$, by C1, $t_i \in \top_i^G(R^*)$ and $s_i \in \beta(t_i)$. Thus, R^* is a rationalizable set in $\top^G$.

Let $s \in \times_{i\in N} R_i^*$ where $R_i^* \equiv \{s_i |\ s \in R^*\}$. Then, for every player i, there exists $t_i \in \top_i^G(R^*)$ such that $s_i \in \beta(t_i)$. Since $R^* \subseteq \times_{i\in N} R_i^*$, again by C1, $t_i \in \top_i^G(\times_{i\in N} R_i^*)$ and $s_i \in \beta(t_i)$. That is, $\times_{i\in N} R_i^*$ is a rationalizable set in $\top^G$. Since $R^* \equiv \cup_{R \text{ is rationalizable in } \top^G} R$, it must be the case that $R^* = \times_{i\in N} R_i^*$. Finally, by C2, the singleton set of a Nash equilibrium in G is a rationalizable set in $\top^G$ and, hence, R^* contains all the Nash equilibria in G. $\square$

Remark. For $Z \subseteq S$ let $\varphi(Z) = \times_{i \in N} \{s_i | \ s_i \in \beta(t_i) \text{ for some } t_i \in \top_i^G(Z)\}$. Then R^* is the largest fixed point of $\varphi : 2^S \mapsto 2^S$. See Apt and Zvesper (2010) and Luo (2001, Sec. 4.1) for a general approach to rationalizable-like solution concepts by using Tarski's fixed point theorem on complete lattices; cf. also Brandenburger et al. (2011) for related discussions. Note that the set of rationalizable strategy profiles may be empty in general game situations. Proposition 1 implies that, under C2, R^* is nonempty if G admits a Nash equilibrium.

3 IENBR and rationalizability

In the literature, rationalizability can also be defined as the outcome of an iterated elimination of never-best responses. We employ a transfinite elimination process that can be used for any arbitrary game.[7] Let λ^0 denote the first element in an ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ. For $S'' \subseteq S' \subseteq S$, S' is said to be *reduced to* S'' (denoted by $S' \to S''$) if, $\forall s \in S' \backslash S''$, there exists some player i such that $s_i \notin \beta(t_i)$ for any $t_i \in \top_i^G(S')$.

Definition 2. An *iterated elimination of never-best responses (IENBR)* is a finite, countably infinite, or uncountably infinite family $\{R^\lambda\}_{\lambda \in \Lambda}$ such that $R^{\lambda^0} = S$, $R^\lambda \to R^{\lambda+1}$ (and $R^\lambda = \cap_{\lambda' < \lambda} R^{\lambda'}$ for a limit ordinal λ), and $R^\infty \equiv \cap_{\lambda \in \Lambda} R^\lambda \to R'$ only for $R' = R^\infty$.

The following Proposition 2 states that Definitions 1 and 2 are equivalent in $\top^G$.

Proposition 2. *R^∞ exists and, under C1, $R^\infty = R^*$ for any IENBR procedure $\{R^\lambda\}_{\lambda \in \Lambda}$.*

Proof. We first show that R^∞ exists. For any $S' \subseteq S$, we define the "next

[7] Lipman (1994) demonstrated that, in infinite games, we may need the transfinite induction to analyze the strategic implication of "common knowledge of rationality." See also Chen et al.'s (2007) Example 1 for the reason why we need a transfinite process for iterated deletion of strictly dominated strategies in general games.

elimination" operation ∇ by

$$\nabla[S'] \equiv \left\{s \in S' \middle| \; \exists i \text{ s.t. } s_i \notin \beta(t_i) \text{ for any } t_i \in \top_i^G(S')\right\}.$$

By the well-ordering principle, the power set of S can be well ordered by a linear order; cf., e.g., Aliprantis and Border (2006, Chapter 1). The existence of a maximal reduction using IENBR is assured by the following prominent "fast" IENBR: $R^\infty \equiv \cap_{\lambda\in\Lambda} R^\lambda$ satisfying $R^{\lambda+1} = R^\lambda \backslash \nabla\left[R^\lambda\right]$ (and $R^\lambda = \cap_{\lambda'<\lambda} R^{\lambda'}$ for a limit ordinal λ), where Λ is an ordinal that is order-isomorphic to the power set of S. Note that $\nabla\left[R^\lambda\right] = \varnothing$ implies $\nabla\left[R^{\lambda'}\right] = \varnothing$ for all $\lambda' > \lambda$. By using the fact that a set is never isomorphic to its power set (cf., e.g., Suppes 1972, Chapter 4, Theorem 23), it is easily verified that $R^\infty \to R'$ only for $R^\infty = R'$.

Now, we consider an IENBR procedure $\{R^\lambda\}_{\lambda\in\Lambda}$. By Definition 2, $\forall s \in R^\infty$, every player i has some $t_i \in \top_i^G(R^\infty)$ such that $s_i \in \beta(t_i)$. So R^∞ is a rationalizable set and, hence, $R^\infty \subseteq R^*$. Under C1, by Proposition 1, R^* is a rationalizable set in $\top^G$ and, hence, survives every round of elimination in Definition 2. So $R^* \subseteq R^\infty$. That is, $R^\infty = R^*$ for any IENBR procedure $\{R^\lambda\}_{\lambda\in\Lambda}$. $\Box$

The definition of IENBR procedure does not require the elimination of *all* never-best response strategies in each round of elimination. This flexibility raises a question whether any IENBR procedure results in a unique set of outcomes. Under C1, Proposition 2 implies that IENBR is indeed a well-defined order-independent procedure in general game situations. If, moreover, C2 is satisfied, Propositions 1 and 2 imply that the IENBR procedure yields a nonempty set of outcomes whenever G has a Nash equilibrium.[8]

4 Nash equilibrium and rationalizability

Propositions 1 and 2 show that every Nash equilibrium is a rationalizable strategy profile and, moreover, every Nash equilibrium survives IENBR. However, the follow-

[8] We note that, in the class of games where strategy sets are compact and payoff functions are continuous, the (countable-round) IENBR procedure results in a nonempty set of outcomes; cf. Dufwenberg and Stegeman (2002, Theorem 1) for the IESDS procedure.

ing example taken from Chen et al. (2007) demonstrates that a Nash equilibrium in the reduced game after an IENBR procedure may be a spurious Nash equilibrium, i.e., it is not a Nash equilibrium in the original game.

Example 2. Consider a two-person symmetric game: $G \equiv \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, and for all $s_i, s_j \in [0, 1]$, $i, j = 1, 2$, and $i \neq j$

$$u_i(s_i, s_j) = \begin{cases} 1, & \text{if } s_i \in [1/2, 1] \text{ and } s_j \in [1/2, 1], \\ 1 + s_i, & \text{if } s_i \in [0, 1/2) \text{ and } s_j \in (2/3, 5/6), \\ s_i, & \text{otherwise.} \end{cases}$$

We consider the standard SEU model for G. It is easily verified that $R^\infty = [1/2, 1] \times [1/2, 1]$ since every strategy $s_i \in [0, 1/2)$ is strictly dominated and hence never a best response. That is, IENBR leaves the reduced game $G|_{R^\infty} \equiv \left(N, \{R_i^\infty\}_{i\in N}, \{u_i|_{R^\infty}\}_{i\in N}\right)$ that cannot be further reduced. Clearly, R^∞ is the set of Nash equilibria in the reduced game $G|_{R^\infty}$, but the set of Nash equilibria in game G is $\{s \in R^\infty |\ s_1, s_2 \notin (2/3, 5/6)\}$. Thus, IENBR generates spurious Nash equilibria $s \in R^\infty$ where some $s_i \in (2/3, 5/6)$. The game of this example is in the class of Reny's (1999) better-reply secure games. Observe that in this game, $u_i(., s_j)$ has a maximizer for $s_j \notin (2/3, 5/6)$, but $u_i(., s_j)$ has no maximizer for $s_j \in (2/3, 5/6)$. That is, some player has no best response in such a spurious Nash equilibrium, while each player should have a best response in a (normal) Nash equilibrium.

For subset $S' \subseteq S$, we say that $G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ has *well-defined best responses on* S' if $\forall i$ and $\forall s \in S'$, $\beta(\delta_i(s)) \neq \varnothing$. Let NE denote the set of Nash equilibria in G, and $NE|_{R^\infty}$ the set of Nash equilibria in the reduced game $G|_{R^\infty} \equiv (N, \{R_i^\infty\}_{i\in N}, \{u_i|_{R^\infty}\}_{i\in N})$. A sufficient and necessary condition under which IENBR generates no spurious Nash equilibria is provided as follows.

Proposition 3. *Under C1 and C2, $NE = NE|_{R^\infty}$ iff G has well-defined best responses on $NE|_{R^\infty}$.*

Proof. ("Only if" part.) Let $s^* \in NE|_{R^\infty}$. Since $NE|_{R^\infty} = NE$, $s_i^* \in \beta(\delta_i(s^*))\ \forall i$. Thus, $\beta(\delta_i(s^*)) \neq \varnothing$ for all i.

("If" part.) (i) Let $s^* \in NE$. Under C1 and C2, by Propositions 1 and 2, $s^* \in R^\infty$ and, hence, $s^* \in NE|_{R^\infty}$. So $NE \subseteq NE|_{R^\infty}$. (ii) Let $s^* \in NE|_{R^\infty}$. Since G has well-defined best responses on $NE|_{R^\infty}$, for every player i there exists some $\hat{s}_i \in S_i$ such that $\hat{s}_i \in \beta\left(\delta_i(s^*)\right)$, which implies that $\hat{s}_i \succsim_{\delta_i(s^*)} s_i^*$ and $(\hat{s}_i, s_{-i}^*) \in R^\infty$ under C1 and C2. Since $s^* \in NE|_{R^\infty}$, $s_i^* \succsim_{\delta_i(s^*)} \hat{s}_i$. Therefore, $s_i^* \sim_{\delta_i(s^*)} \hat{s}_i$ and, hence, $s_i^* \in \beta\left(\delta_i(s^*)\right)$. That is, $s^* \in NE$. So $NE|_{R^\infty} \subseteq NE$. $\square$

This sufficient and necessary condition in Proposition 3 does not involve any topological assumption on the original or the reduced games. In Chen et al.'s (2007) Corollary 4, some classes of games with special topological structures were proved to "preserve Nash equilibria" for the iterated elimination of strictly dominated strategies. These results are also applicable to the IENBR procedure defined in this paper. If a game is solvable by an IENBR procedure, the following corollary asserts that the unique strategy profile surviving the procedure is the only Nash equilibrium.

Corollary 1. *Under C1 and C2, $R^\infty = NE$ if $|R^\infty| = 1$.*

Proof. Let $R^\infty = \{s^*\}$. By C2, s_i^* is a best response to $\delta_i(s^*)$ for every player i. So $s^* \in NE$ and hence $R^\infty \subseteq NE$. By Proposition 1, $NE \subseteq R^\infty$. $\square$

5 Epistemic conditions of rationalizability

In this section we provide epistemic conditions for rationalizability in general games. In doing epistemic analysis, we need to extend the model of type in Section 2 to the space of states. Consider a space Ω of *states*, with typical element $\omega \in \Omega$. A subset $E \subseteq \Omega$ is referred to as an *event*. A *model of type on* Ω is given by

$$\top \equiv \{\top_i(\cdot)\}_{i \in N},$$

where $\top_i(\cdot)$ is defined over (nonempty) subsets $E \subseteq \Omega$. The set $\top_i(E)$ is player i's type space for given event E, which can be interpreted as player i's type space conditional on event E; each type $t_i \in \top_i(E)$ has a preference relation $\succsim_{t_i}$ defined on player i's strategies in S_i under which the complement of E is regarded as impossible.

For example, if $\top_i(E)$ is applied to the case of the probability measure space, $\top_i(E)$ can be considered as the space of probability measures conditional on subset E of Ω. The model of type on Ω can also be viewed as a type structure used in epistemic game theory to model interactive beliefs in which a type of a player is a joint belief about the states of nature and the types of the other players (see, e.g., Brandenburger (2007)).

An *epistemic model for* $\top^G$ is defined by

$$\mathcal{M}\left(\top^G\right) \equiv \left(\Omega,\ \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}\right),$$

where Ω is the space of states, $\top$ is a model of type on Ω, $\mathbf{s}_i(\omega) \in S_i$ is player i's strategy at state ω, and $\mathbf{t}_i(\omega) \in \top_i(\Omega)$ is player i's type at state ω; cf., e.g., Aumann (1999) and Osborne and Rubinstein (1994, Chapter 5).[9] Denote by $\mathbf{s}(\omega)$ the strategy profile at ω and let

$$S^E \equiv \{\mathbf{s}(\omega) \mid \omega \in E\}.$$

Apparently, from an analyst's point of view, the model of type, $\top$, defined on Ω should be consistent with the model of type, $\top^G$, defined on G. For this purpose, in this paper we require the epistemic model $\mathcal{M}\left(\top^G\right)$ to satisfy the following consistency property:

[Consistency] For any event $E \subseteq \Omega$, $\top_i(E) = \top_i^G(S^E)\ \forall i$.

That is, the consistency property requires that the type space on an event be consistent with the type space on the strategies projected from the event and, thus,

[9] We take a point of view that an epistemic model is a pragmatic and convenient framework to be used for doing epistemic analysis; cf. Aumann and Brandenburger (1995, Sec. 7a) for related discussions. Mertens and Zamir (1985) constructed a well-defined compact Hausdorff space of types in a probabilistic setting, Heifetz and Samet (1998) provided an alternative "topology-free" construction of type space, and Epstein and Wang (1996) constructed a well-defined compact Hausdorff space of types in a more general setting of regular preferences; see Epstein (1997) for various "models of preference." Di Tillio (2008) constructed a well-defined space of types under some mild assumptions on preferences (reflexivity, transitivity, and monotone continuity) in finite strategic games. In this paper, we are mainly concerned with the analysis of the game-theoretic solution concept of rationalizability in general situations. In particular, we do not assume that preferences have utility function representations.

each player behaves in a natural way with respect to the marginalization in the epistemic model. This requirement is much in the same spirit of the notion of "coherence" imposed in the analysis of hierarchy of beliefs and preferences (see, e.g., Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Epstein and Wang (1996)). We would like to point out that the consistency property is not a behavioral condition for the players in games, but it is made only for the (comprehensive) epistemic model adopted by an analyst to be harmonious with the (simple) analytical framework used in Section 2.

We say "*player i knows/believes an event E at ω*" if $\mathbf{t}_i(\omega) \in \top_i(E)$ since the complement of E is regarded as impossible under $\mathbf{t}_i(\omega) \in \top_i(E)$.[10] Let

$$K_i E \equiv \{\omega \in \Omega |\ i \text{ knows } E \text{ at } \omega\}.$$

An event $\boxed{E} \subseteq E$ is called a *common-knowledge/self-evident event (in E)*, if

$$\boxed{E} \subseteq K_i \boxed{E} \text{ for all } i \in N.$$

Say player i is "*rational at ω*" if $\mathbf{s}_i(\omega)$ is one of most preferred actions for $\mathbf{t}_i(\omega)$. Let

$$\mathfrak{R}_i \equiv \{\omega \in \Omega |\ i \text{ is rational at } \omega\}$$

and

$$\mathfrak{R} \equiv \cap_{i \in N} \mathfrak{R}_i.$$

That is, $\mathfrak{R}$ is the event "everyone is rational." The following Proposition 4 provides an epistemic characterization for rationalizability: the notion of rationalizability is the strategic implication of common knowledge of rationality.

Proposition 4. *(1) In any epistemic model $\mathcal{M}\left(\top^G\right)$, $S^{\boxed{\mathfrak{R}}}$ is a rationalizable set in $\top^G$. (2) Suppose that R is a rationalizable set in $\top^G$. Then, there is an epistemic model $\mathcal{M}\left(\top^G\right)$ in which $S^{\boxed{\mathfrak{R}}} = R$ for some common-knowledge event $\boxed{\mathfrak{R}}$.*

[10] This formalism can be easily applied to Aumann's definition of knowledge by the possibility correspondence in a semantic framework and the notion of "belief with probability one" in a probabilistic model, for instance.

Proof. (1) Since $\boxed{\mathfrak{R}} \subseteq \mathfrak{R}$ is a common-knowledge event, for any $\omega \in \boxed{\mathfrak{R}}$, $\mathbf{s}_i(\omega) \in \beta(\mathbf{t}_i(\omega))$ and $\mathbf{t}_i(\omega) \in \top_i\left(\boxed{\mathfrak{R}}\right)$ for all i. By Consistency, $\mathbf{t}_i(\omega) \in \top_i^G\left(S^{\boxed{\mathfrak{R}}}\right)$. Therefore, $\forall i$ and $\forall s \in S^{\boxed{\mathfrak{R}}}$, there exists some $t_i \in \top_i^G(S^{\boxed{\mathfrak{R}}})$ such that $s_i \in \beta(t_i)$. That is, $S^{\boxed{\mathfrak{R}}}$ is a rationalizable set in $\top^G$.

(2) Let R be a rationalizable set in $\top^G$. Define an epistemic model for $\top^G$:

$$\mathcal{M}\left(\top^G\right) \equiv \left(\Omega,\ \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}\right),$$

such that

(i) $\Omega = \left\{(s_i, t_i)_{i\in N} \mid t_i \in \top_i^G(R) \text{ and } s_i \in \beta(t_i) \cap R_i\right\}$;
(ii) $\forall i$, $\top_i(E) = \top_i^G(S^E)$ if $E \subseteq \Omega$;
(iii) $\forall i$, $\mathbf{s}_i(\omega) = s_i$ and $\mathbf{t}_i(\omega) = t_i$ if $\omega = (s_i, t_i)_{i\in N}$.

Clearly, every player i is rational across states in Ω. By Consistency, $\Omega \subseteq K\Omega$. Therefore, $\boxed{\mathfrak{R}} = \Omega$ is common-knowledge event satisfying $S^\Omega = R$. $\square$

Remark. In the standard semantic model of knowledge, it is well known that the above "fixed-point" definition of "common knowledge" is equivalent to the traditional "iterative" formalism of "common knowledge;" see, e.g., Aumann (1976, 1999) and Monderer and Samet (1989). In general cases, the "fixed-point" definition of "common knowledge" is a more fundamental notion. Under the "monotonic" information and knowledge structures, it can be shown that the "fixed-point" definition of "common knowledge" is equivalent to an "iterative" notion of "common knowledge" *possibly by using transfinite levels of mutual knowledge*; see Heifetz (1996, 1999) for more discussions. If, moreover, the information and knowledge structures satisfy a "limit closure" property: what happens at finite levels determines what happens at the limit, it can be shown that the "fixed-point" definition of "common knowledge" is equivalent to the traditional "iterative" definition *by using a countably infinite number of levels of mutual knowledge*; cf. Fagin et al. (1999).

Within the standard expected utility framework in finite games, Brandenburger and Dekel (1987) offered the notion of "a posteriori equilibrium," a strengthening of Aumann's (1974) notion of subjective correlated equilibrium, and showed an equivalence between rationalizability and a posteriori equilibrium. The equivalence implies that the assumption of common knowledge of rationality also provides a formal epistemic justification for this equilibrium notion. In finite models, Epstein (1997) extended this equivalence result to more general "regular" preferences including the subjective expected utility model. We end this section by presenting such an equivalence result for arbitrary games with various modes of behavior in the analytical framework used in this paper.

A strategy-profile specification function $\mathbf{s} : \Omega \to S$ in an epistemic model $\mathcal{M}\left(\top^G\right)$ for game G is said to be an *a posteriori equilibrium in* $\mathcal{M}\left(\top^G\right)$ if for every player $i \in N$,[11]

$$\forall \omega \in \Omega,\ \mathbf{s}_i\left(\omega\right) \succsim_{\mathbf{t}_i(\omega)} s_i\ \forall s_i \in S_i,$$

i.e., $\mathbf{s}_i\left(\omega\right) \in \beta\left(\mathbf{t}_i\left(\omega\right)\right)$.

Proposition 5. *The strategy profile s^* is rationalizable in $\top^G$ if and only if there exist an epistemic model $\mathcal{M}\left(\top^G\right)$ and an a posteriori equilibrium* $\mathbf{s}$ *in $\mathcal{M}\left(\top^G\right)$ such that $s^* = \mathbf{s}\left(\omega\right)$ for some $\omega \in \Omega$.*

Proof. "if" part: Let $\mathbf{s}$ be an a posteriori equilibrium in an epistemic model $\mathcal{M}\left(\top^G\right)$. Then, for every player i and every $s \in S^{\Omega}$, $s_i \in \beta(t_i)$ for some $t_i \in \top_i(\Omega)$. By Consistency, for every player i and every $s \in S^{\Omega}$, $s_i \in \beta(t_i)$ for some $t_i \in \top_i^G(S^{\Omega})$. That is, the set S^{Ω} is a rationalizable set in $\top^G$. Thus, the profile s^* is rationalizable in $\top^G$ if $s^* = \mathbf{s}\left(\omega\right)$ for some $\omega \in \Omega$.

"Only if" part: Let s^* be a rationalizable strategy profile in $\top^G$. Then, there is a rationalizable set R in $\top^G$ which contains s^*. Thus, for every player i and every

[11] This definition of "a posteriori equilibrium" does not involve an exogenous informational partition for each player as in Bandenburger and Dekel (1987) and Epstein (1997). However, an endogenous (possibly non-partitional) informational structure for each player can be elicited from the player's preference relation at states; cf. Morris (1996) and Chen and Luo (2011b) for more discussions.

$s \in R$, there is $t_i \in \top_i^G(R)$ such that $s_i \in \beta(t_i)$. Define an epistemic model for G :

$$\mathcal{M}\left(\top^G\right) \equiv \left(\Omega,\ \top, \{\mathbf{s}_i\}_{i\in N}, \{\mathbf{t}_i\}_{i\in N}\right),$$

such that

$$\begin{array}{l}\text{(i) } \Omega = \left\{(s_i, t_i)_{i\in N} \mid t_i \in \top_i^G(R) \text{ and } s_i \in \beta(t_i)\right\};\\ \text{(ii) } \forall i,\ \top_i(E) = \top_i^G(S^E) \text{ if } E \subseteq \Omega;\\ \text{(iii) } \forall i,\ \mathbf{s}_i(\omega) = s_i \text{ and } \mathbf{t}_i(\omega) = t_i \text{ if } \omega = (s_i, t_i)_{i\in N}.\end{array}$$

Therefore, for every player i and every $\omega = (s_i, t_i)_{i\in N}$ in Ω, $\mathbf{s}_i(\omega) \in \beta(\mathbf{t}_i(\omega))$. That is, $\mathbf{s}$ is an a posteriori equilibrium in $\mathcal{M}\left(\top^G\right)$. Thus, for each rationalizable profile s^* in $\top^G$, we can find an a posteriori equilibrium $\mathbf{s}$ in $\mathcal{M}\left(\top^G\right)$ and a state $\omega^* = (s_i^*, t_i^*)_{i\in N}$ in Ω such that $s^* = \mathbf{s}(\omega^*)$. $\square$

6 Rationalizability and iterated strict dominance

In this section, we show that, through examples, rationalizability in general game situations neither implies nor is implied by iterated strict dominance defined by Chen et al. (2007). This is because, in the general environments, an undominated strategy need not be a best response in a model of type, and conversely, a best response in a model of type is not necessarily undominated, even in the case of (correlated) probabilistic models.[12]

Since the model of type can be applied to some particular class of probabilistic models such as the product (independent) probability model and the degenerated (point) probability model, it is easy to see that an undominated strategy may fail to be a best response in finite games with such restrictive types of probabilistic beliefs; cf., e.g., Brandenburger and Dekel (1987, Sec. 3). Alternatively, the following Example 3 (due to Andrew Postlewaite), which appears in Bergemann and Morris (2005a, Footnote 8), shows that an undominated strategy need not be a best response in an infinite game with (correlated) probabilistic beliefs.[13]

[12] Chen and Luo (2011a) showed that if the sets of strategies are compact Hausdorff spaces and the payoff functions are continuous and satisfy a condition called "concave-like," then a strategy is undominated if and only if it is a best response in the standard SEU model.

[13] For simplicity, we here consider strategies dominated by pure strategies. Examples 3 and 4 are still valid if we allow for using mixed strategies.

Example 3. Consider a two-person symmetric game $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1,2\}$, and for $i = 1,2$, $S_i = \{0,1,2,...\}$ and

$$u_i(s_i, s_{-i}) = \begin{cases} 1, & \text{if } s_i = 0; \\ 2, & \text{if } s_i \geq 1 \text{ and } s_i > s_{-i}; \\ 0, & \text{if } s_i \geq 1 \text{ and } s_i \leq s_{-i}. \end{cases}$$

Let $\top^G$ be the model of type generated by expected utility preferences with (countably additive) probability measures, i.e., $\top_i^G(S') = \Delta\left(S'_{-i}\right)$ for all $S' \subseteq S$. Clearly, $s_i = 0$ is not strictly dominated, because for any $s_i \geq 1$, $u_i(0, s_{-i}) = 1 > 0 = u_i(s_i, s_{-i})$ for $s_{-i} \geq s_i$. But, $s_i = 0$ cannot be a best response in $\top^G$. To see this, note that for any $\mu_i \in \Delta(S_{-i})$ and any $s_i > 0$,

$$\int u_i(s_i, s_{-i})\, d\mu_i(s_{-i}) = 2\mu_i(\{s_{-i} |\ s_{-i} < s_i\}) \to 2 \text{ as } s_i \to \infty.$$

Hence, there is some $s_i > 0$ such that $\int u_i(s_i, s_{-i})\, d\mu_i(s_{-i}) > 1 = \int u_i(0, s_{-i})\, d\mu_i(s_{-i})$.

The following Example 4, which is modified from Stinchcombe (1997), shows that a strictly dominated strategy can be a best response in a game with "finitely additive" probabilistic beliefs.[14]

Example 4. Consider a two-person game $G = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1,2\}$, $S_1 = \{0,1\}$, $S_2 = \{1,2,...\}$, and $u_1(0, s_2) = 0$, $u_1(1, s_2) = 1/s_2$, $u_2(s_2, 0) = 0$, and $u_2(s_2, 1) = s_2$ for all $s_2 \in S_2$. Let the algebra on S_2 be the power set of S_2. Let $\top^G$ be the model of type generated by expected utility preferences generated by *finitely additive* probability charges, i.e., $\top_i^G(S') = ba\left(S'_{-i}\right)$ for all $S' \subseteq S$, where $ba\left(S'_{-i}\right)$ is the space of finitely additive probability charges on S'_{-i}.

Clearly, $s_1 = 0$ is strictly dominated by $s_1 = 1$. However, $s_1 = 0$ can be a best response in $\top^G$. To see this, it suffices to show that $\int u_1(1, s_2)\, d\mu(s_2) = \int u_1(0, s_2)\, d\mu(s_2) = 0$ for some $\mu \in ba(S_2)$. To find such a μ, let $\mu_m \in ba(S_2)$ be

[14] A similar example can also be found in Adams (1962), Seidenfeld and Schervish (1983), and Wakker (1993).

the uniform distribution on $\{1, 2, ..., m\}$. By Alaoglu's Theorem (see Royden (1968, Theorem 10.17)), there are $\mu \in ba(S_2)$ and a sequence of $\{\mu_m\}$ such that

$$\lim_{m\to\infty} \mu_m(E) = \mu(E) \text{ for each } E \subseteq \mathbb{N}. \tag{1}$$

Since $u_1(0, s_2) = 0$ for all s_2, it follows that $\int u_1(0, s_2)\, d\mu(s_2) = 0$. To see $\int u_1(1, s_2)\, d\mu(s_2) = 0$, observe that for every $K \geq 1$,

$$0 \leq \int u_1(1, s_2)\, d\mu(s_2) \leq \sum_{s_2=1}^{K} \mu(\{s_2\}) + \frac{1}{K}\mu(\{s_2 : s_2 > K\}) . \tag{2}$$

Since $\mu_m(\{s_2\}) \to 0$ and $\mu_m(\{s_2 : s_2 > K\}) \to 1$, it follows from (1) that $\mu(\{s_2\}) = 0$ and $\mu(\{s_2 : s_2 > K\}) = 1$. Since (2) holds for all $K \geq 1$, $\int u_1(1, s_2)\, d\mu(s_2) = 0$.

A strategy $s_i \in S_i$ is said to be *dominated given* $S' \subseteq S$ if for some strategy $\widehat{s}_i \in S_i$, $u_i(\widehat{s}_i, s'_{-i}) > u_i(s_i, s'_{-i})$ for all $s'_{-i} \in S'_{-i}$. For any subsets $S', S'' \subseteq S$ where $S'' \subseteq S'$, we use the notation $S' \mapsto S''$ to signify that for any $s \in S' \backslash S''$, some s_i is dominated given S'. Let λ^0 denote the *first* element in an *ordinal* Λ, and let $\lambda + 1$ denote the *successor* to λ in Λ. In general games, Chen et al. (2007) offered the well-defined order-independent iterated elimination of strictly dominated strategies:

Definition 3. An *iterated elimination of strictly dominated strategies (IESDS*)* is defined as a finite, countably infinite, or uncountably infinite family $\left\{\mathcal{D}^{\lambda}\right\}_{\lambda\in\Lambda}$ such that $\mathcal{D}^{\lambda^0} = S$ (and $\mathcal{D}^{\lambda} = \cap_{\lambda'<\lambda}\mathcal{D}^{\lambda'}$ for a limit ordinal λ), $\mathcal{D}^{\lambda} \mapsto \mathcal{D}^{\lambda+1}$, and $\mathcal{D} \equiv \cap_{\lambda\in\Lambda}\mathcal{D}^{\lambda} \mapsto \mathcal{D}'$ only for $\mathcal{D}' = \mathcal{D}$.

Next, we present an equivalence result between rationalizability and IESDS* in the class of dominance-solvable games. We say that a game G is "dominance-solvable" if the procedure of IESDS* leads to a unique strategy profile – i.e., by performing the procedure of iterated elimination of strictly dominated strategies, there is only one strategy left for each player; for example, the standard Cournot game (Moulin, 1984), Bertrand oligopoly with differentiated products, and the arms-race games (Milgrom and Roberts, 1990). We need the following condition on a type model $\top^G$.

C3 (Strong Monotonicity) If a strategy $\widehat{s}_i \in S_i$ strictly dominates another strategy $s_i \in S_i$ given S' – i.e. $u_i\left(\widehat{s}_i, s'_{-i}\right) > u_i\left(s_i, s'_{-i}\right)\ \forall s'_{-i} \in S'_{-i}$, then $\widehat{s}_i \succ_{t_i} s_i$ for all $t_i \in \top_i^G(S')$.

The Strong Monotonicity requires that a strategy be strictly preferred to another strategy if the former strategy strictly payoff-dominates the latter one. This condition on $\top^G$ seems to be rather natural, and is satisfied by most of preference models discussed in the literature, e.g., the SEU model (Savage 1954), the OEU model (Borgers 1993), the probabilistic sophistication model (Machina and Schmeidler 1992), the multi-priors model (Gilboa and Schmeidler 1989), the Choquet expected utility model (Schmeidler 1989), the lexicographic preference model (Blume et al. 1991), the Knightian uncertainty model (Bewley 1986), and so on.[15] From a decision-theoretic point of view, the "transitivity" or "strong monotonicity" condition is considered to be more basic tenets of rationality than the Sure-Thing-Principle and other components of the standard Savage model; see Luce and Raiffa (1957, Chapter 13) and Epstein (1997) for more discussions. The following Proposition 6 asserts that in dominance-solvable games, the notion of rationalizability defined in any type model $\top^G$ satisfying Diracability and Strong Monotonicity (but not satisfying Monotonicity) is equivalent to the Nash equilibrium, which can be solved by IESDS*.

Proposition 6. *Suppose that G is a dominance-solvable game with a type model $\top^G$ satisfying C2 and C3. Then, $\mathcal{D} = R^* = NE$.*

Proof. Since G is dominance-solvable, $\mathcal{D} = NE$. Let R be a rationalizable set in $\top^G$. Then, by C3, R is an undominated set – i.e., for every i, $s_i \in R_i$ is not dominated given R. Therefore, $R \subseteq \mathcal{D}^\lambda$ for all λ and, hence, $R^* \subseteq \mathcal{D} = NE$. By C2, the singleton of a Nash equilibrium is a rationalizable set in $\top^G$. Consequently, $R^* = \mathcal{D} = NE$. $\square$

[15] Nevertheless, as demonstrated in Example 4, the expected utility preference model with a finitely additive probability charge may violate C3.

Remark. Proposition 6 says that in dominance-solvable games, rationalizability defined in any type model $\top^G$ satisfying C2 and C3 yields the unique set of outcomes of iterated strict dominance, which is consistent with the Nash equilibrium outcome. The result implies that the Nash equilibrium behavior is observationally indistinguishable from the rationalizable strategic behavior in such type models. In the classical Cournot duopoly game, for example, the Cournot-Nash outcome is robust to the rationalizable strategic behavior in a wide range of type models including the SEU model, the OEU model, the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the lexicographic preference model, the Knightian uncertainty model, etc. Proposition 6 also implies that IENBR and IESDS* generate no spurious Nash equilibria in dominance-solvable games. We would like to emphasize that Proposition 6 can be used for a type model, without imposing C1, where the notion of rationalizability may fail to have properties in Propositions 1-2.

7 Concluding remarks

In this paper, we have presented a simple and unified framework for analyzing rationalizable strategic behavior in general environments – i.e., arbitrary strategic games with various modes of behavior; in particular, we have introduced the "model of type" to define the notion of rationalizability in games with (in)finite players, arbitrary strategy spaces, and arbitrary payoff functions. We have investigated properties about rationalizability in general situations and shown that the notion of rationalizability possesses nice properties similar to those in finite games. More specifically, under the mild condition C1, we have shown the following results on rationalizability in general situations:

- The union of all the rationalizable sets is the largest rationalizable set in product form (Proposition 1).
- The largest rationalizable set can be derived by the (possibly transfinite) iterated elimination process of IENBR; IENBR is a well-defined

order-independent procedure (Proposition 2).

Within a standard epistemic model, we have formulated and established the following characterizations for rationalizability in general situations:

- The notion of rationalizability can be characterized by common knowledge of rationality (Proposition 4).
- The notion of rationalizability is equivalent to the notion of a posteriori equilibrium (Proposition 5).

Our approach in this paper is completely topology-free, and is applicable to any arbitrary strategic game. We have demonstrated that, through examples, rationalizability is in general not equivalent to iterated strict dominance in general game situations. However, in dominance-solvable games, the rationalizable strategic behavior in a wide range of preference models yields the set of outcomes of iterated strict dominance, which coincides with the Nash equilibrium outcome (Proposition 6).[16]

In this paper, we have also investigated, under C2, the relationship between rationalizability and Nash equilibrium in general games. While every Nash equilibrium survives the IENBR procedure, a Nash equilibrium in the final reduced game after IENBR may fail to be a Nash equilibrium in the original game. That is, the IENBR procedure may generate spurious Nash equilibria in infinite games, e.g. Reny's (1999) better-reply secure games. We have provided a sufficient and necessary condition to guarantee no spurious Nash equilibria (Proposition 3); in particular, the unique Nash equilibrium can be obtained by IENBR if the procedure of IENBR yields a singleton outcome (Corollary 1).

We would like to emphasize that one important feature of this paper is that the framework allows the players to have various preferences which include the subjective expected utility as a special case. In the light of the analysis of this

[16] Chen and Luo (2011a) showed that rationalizability under general preferences can be indistinguishable from the outcome of the IESDS procedure for a class of (in)finite games.

paper, we seek fairly natural and few behavioral assumptions on players' preference relations as weak as possible to make our analysis applicable to a wide range of strategic problems; in particular, we do not assume that preferences have utility function representations. The general analysis of this paper is applicable to any arbitrary strategic game with various modes of behavior.[17]

To close this paper, we would like to point out some possible extensions of this paper for future research. In the (finite) Bayesian game model, Dekel et al. (2007) offered the notion of interim correlated rationalizability, and Morris and Takahashi (2011) examined a variant of preference-correlated rationalizability. We note that the framework presented in this paper can be used for a general analysis of rationalizability in incomplete-information environments. For example, we can recast Dekel et al.'s (2007) notion of interim correlated rationalizability by using a framework in which each type of a player is viewed as an independent agent-player and the model of type for each agent-player specifies a subjective probability measure space for each product set of the payoff-relevant states, the other players' types, and possible restriction of action profiles, where each probability distribution over states, the other players' types and actions is consistent with the type's a prior belief about the payoff-relevant states and the other players' types. This framework can also be used for analyzing interim independent rationalizability and preference-correlated rationalizability. The extension of this paper to general games with incomplete information is an important subject for further research. In finite strategic games, Ambrus (2006) and Luo and Yang (2009) offered the notion of (Bayesian) coalitional rationalizability in complex social interactions. The extension of this paper to permit social and coalitional interactions in general situations is an intriguing topic worth further investigation. The exploration of the notion of extensive-form rationalizability in general dynamic games also remains to be an important research topic for further study.

[17]In this paper, C1 is perhaps the only essential behavioral assumption under which the rationalizability defined in general situations possesses nice properties as in the case of finite games. C2 can be removed if one does not care about its relationship with the Nash equilibrium. Morris and Takahashi (2011) did not impose this condition in their analysis. C3 is rather mild and innocuous, and the condition is satisfied by almost all preference models discussed in the literature.

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