Contagion and Uninvadability in Social Networks with Bilingual Option*

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Abstract

We study the long run outcome of local interactions in an infinite population of players, each of whom chooses one of two conventions or adopts both (i.e., chooses the "bilingual option") at an additional cost. In this class of games, we completely characterize when a convention spreads contagiously from a finite subset of players to the entire population in some network, and conversely, when a convention is never invaded by the other convention in any network. Generically, at least one convention spreads contagiously in some network, and for some range of payoff parameters, both conventions each spread contagiously in respective networks. Our proofs for this characterization provide new insights on how the network structure affects contagion. *Journal of Economic Literature* Classification Numbers: C72, C73, D83.

Keywords: equilibrium selection; bilingual game; local interaction; network; contagion; uninvadability.

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Web page: www.oyama.e.u-tokyo.ac.jp/papers/bilingual.html.

1 Introduction

A small group of individuals can influence the long-run outcome in a large population through local interactions. Such a phenomenon is called contagion, also called diffusion or infection interchangeably, and has attracted much attention. Analyzing 2×2 coordination games, Morris (2000) shows that an action can spread contagiously in some network if and only if it is the risk-dominant action, and a simple linear network is more contagion-inducing than any other network in the sense that if contagion occurs in some network, it occurs in the simple linear network. In this paper, we extend his analysis to a particular 3×3 game, which we call the bilingual game. The purpose of this exercise is two-hold: (1) to analyze whether and how the presence of the bilingual option affects the possibility of contagion; (2) to investigate finer structures of networks than what the previous analysis on 2×2 games can reveal.

Consider an infinite population of players who are connected with each other through a graph ("social network"). Suppose that each player uses one of two computer programming languages, or two types of technologies in general, A and B. The payoff from each interaction with his neighbors is given by the following 2×2 coordination game:

	A	B
A	a, a	b, c
B	c, b	d, d

where $a > c$ and $d > b$, so that (A, A) and (B, B) are strict Nash equilibria. We assume that $a > d$, i.e., (A, A) Pareto-dominates (B, B), while $a - c < d - b$, i.e., (B, B) risk-dominates (A, A). We further assume that $d \geq c$ (which together with the above assumptions implies that $a \geq b$), i.e., coordination on some action is always better than miscoordination. It is well known (see, e.g., Morris (2000)) that the risk-dominant action B spreads contagiously from a finite subset of players to the entire population in some network, and that it is never invaded by the other action A in any network. Thus, in 2×2 coordination games, the risk-dominant action is always both contagious and uninvadable. In fact, contagion and uninvadability are equivalent in this class of games.

Now suppose that players can adopt a combination of the two actions, a "bilingual option" AB, with an additional cost $e > 0$. A player who plays AB receives a (gross) payoff a (d, resp.) from an interaction with an A-player (B-player, resp.). When two AB-players interact, they adopt the superior action A and receive a. This situation is described by the following payoff matrix:[1]

[1] This game has been studied by Galesloot and Goyal (1997), Goyal and Janssen (1997),

	A	AB	B
A	a, a	$a, a - e$	b, c
AB	$a - e, a$	$a - e, a - e$	$d - e, d$
B	c, b	$d, d - e$	d, d

where (A, A) and (B, B) are the only pure-strategy Nash equilibria. One may expect that, when the value of the cost parameter e is large, the action AB is not much relevant so that the situation is close to the previous 2×2 case, while as e becomes smaller, AB becomes closer to dominating B so that eventually B will be abandoned and only A will survive.

In this paper, we completely characterize when an action is contagious and when it is uninvadable in this class of 3×3 games. Conforming to the conjecture in the previous paragraph, we show that if e is large, then B is contagious and uninvadable, while if e is small, then A is contagious and uninvadable. Generically, either A or B is contagious, but, in contrast to the 2×2 case, both actions are each contagious if e is in a medium range (which is nonempty and open under an additional condition on parameter values), i.e., A spreads contagiously in some networks while B does in some others. In other words, uninvadability is a strictly stronger property than contagion.

Our proofs for the above characterization provide new insights on how the network structure affects contagion. A class of networks is called *critical* if these networks induce all possible contagion, i.e., whenever an action can spread contagiously in some network, it does so within this class of networks. We show that the class of all "linear" networks is not critical in determining contagion in the bilingual game, and provide an example of a critical class that includes "non-linear" networks.

We also ask a comparative question: which network is more likely to induce contagion? We say that a network is *more contagion-inducing* than another network if any action that is contagious in the latter network is also contagious in the former network. This preorder is incomplete, unlike the "contagion threshold" that characterizes contagion for 2×2 coordination games (Morris (2000)). We introduce the notion of weight-preserving node identification between two networks, and show that this notion provides a sufficient condition for a network to induce more contagion than another.

In his series of papers, Morris (1997, 1999, 2000) defines general notions of contagion and uninvadability, develops a method using potential functions to provide a sufficient condition for uninvadability (and hence a necessary condition for contagion), and gives an example of a symmetric 4×4 game to demonstrate the multiplicity of contagious actions. In particular, in his 4×4 example, contagion of these actions occurs in "linear" networks. For

Immorlica et al. (2007), and Easley and Kleinberg (2010).

our class of games, we utilize the potential method to show uninvadability, while we construct a "non-linear" network to obtain contagion of one of the two actions.

Contagious behavior in the bilingual game is analyzed by Goyal and Janssen (1997) and Immorlica et al. (2007). Both papers, however, focus on specific classes of networks and provide only sufficient (necessary, resp.) conditions for contagion (uninvadability, resp.), which are strictly stronger than the condition we obtain as a full characterization. This implies that their classes of networks are not critical.

As Morris (1997, 1999) argues, local interaction games and incomplete information games have formal connections, and both belong to a more general class of "interaction games". Accordingly, our results on local interaction games can be interpreted in the context of incomplete information games, whereby we provide interesting implications on global games and robustness to incomplete information.

2 Local Interaction Games

Let $\mathcal{X}$ be a countably infinite set of players, and $P\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ a function such that

1. $P(x,x)=0$ for all $x\in\mathcal{X}$,
2. $P(x,y)=P(y,x)$ for all $x,y\in\mathcal{X}$, and
3. $0<\sum_{y\in\mathcal{X}}P(x,y)<\infty$ for all $x\in\mathcal{X}$.

A *local interaction system*, or *network*, $(\mathcal{X},P)$ defines an undirected graph with vertices $\mathcal{X}$ and edges weighted by P.[2] (We will use the terms "local interaction system" and "network" interchangeably.) We will restrict our attention to *unbounded* local interaction systems; i.e., $\sum_{(x,y)\in\mathcal{X}\times\mathcal{X}}P(x,y)=\infty$. Write $\Gamma(x)=\{y\in\mathcal{X}\mid P(x,y)>0\}$ for the set of neighbors of player $x\in\mathcal{X}$. Denote

$$P(y|x)=\frac{P(x,y)}{\sum_{y'\in\Gamma(x)}P(x,y')},$$

which is well defined due to property 3 above.

Players have a (common) finite set of actions S and a (common) payoff function $u\colon S\times S\to\mathbb{R}$. With the action set S fixed, a *local interaction game* is represented by the tuple $(\mathcal{X},P,u)$. Let $\Delta(S)$ denote the set of probability

[2] One could instead focus on local interaction systems with *constant weights*, where $P(x,y)\in\{0,1\}$ for all $x,y\in\mathcal{X}$. All the results in this paper would remain unchanged since any local interaction system with rational weights can be replicated by a local interaction system with constant weights.

distributions over S. Given payoff function u, write $br(\pi)$ for the set of pure best responses to $\pi \in \Delta(S)$:

$$br(\pi) = \{h \in S \mid u(h,\pi) \geq u(h',\pi) \text{ for all } h' \in S\}, \tag{2.1}$$

where $u(h,\pi) = \sum_{k\in S} \pi_k u(h,k)$.

An *action configuration* is a function $\sigma\colon \mathcal{X} \to S$. Given an action configuration σ, we denote by $\pi(\sigma|x) \in \Delta(S)$ the action distribution, weighted by $P(\cdot|x)$, over the actions of player x's neighbors: i.e.,

$$\pi_h(\sigma|x) = \sum_{y\in\Gamma(x)\colon \sigma(y)=h} P(y|x).$$

The payoff for player $x \in \mathcal{X}$ playing action $s \in S$ is given by the weighted sum (with respect to $P(\cdot|x)$) of payoffs from the interactions with his neighbors:

$$U(s,\sigma|x) = \sum_{y\in\Gamma(x)} P(y|x)u(s,\sigma(y)),$$

which equals $u(s,\pi(\sigma|x))$. Write $BR(\sigma|x)$ for the set of pure best responses for player x to action configuration σ:

$$BR(\sigma|x) = \{s \in S \mid U(s,\sigma|x) \geq U(s',\sigma|x) \text{ for all } s' \in S\}, \tag{2.2}$$

which equals $br(\pi(\sigma|x))$.

We consider the sequential best response dynamics on network $(\mathcal{X}, P)$ as defined below. (There being finitely many actions, for a sequence of actions $(s^t)_{t=0}^{\infty}$, $\lim_{t\to\infty} s^t = s$ if and only if there exists T such that $s^t = s$ for all $t \geq T$.)

Definition 1. A sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *best response sequence* if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in \mathcal{X}$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if there exists $T \geq 0$ such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then $\lim_{t\to\infty} \sigma^t(x) \neq s$.

Property (i) requires that in each period at most one player revise his action,[3,4] while property (ii) requires that the revising player switch to a myopic best response to the current distribution of his neighbors' actions. Property (iii) requires that actions that are never a best response be abandoned eventually. In particular, (ii) and (iii) imply that if there exists T such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then there exists T' such that $\sigma^t(x) \neq s$

[3] In continuous time models, this assumption would be replaced by the one that action revision timings follow Poisson processes independent among players.

[4] While Proposition 2 will rely on this assumption, all the other results would hold also under the simultaneous best response dynamics and others. See the discussion after Definition 2 and Appendix A.1.

for all $t \geq T'$. Note that for a given initial action configuration, there are in general multiple best response sequences, as properties (i) and (iii) do not specify which player revises actions in which period.

We are concerned with the following questions. Is it possible in some network and some finite group of players such that if that group initially plays action s^*, then the whole population will eventually play s^*? In this case, s^* is said to be contagious. Or, is it always the case in any network that if s^* is played by almost all players, it continues to be played by almost all players? If so, s^* is said to be uninvadable. Below we formally define the relevant concepts following Morris (1997, 1999).

Definition 2. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *contagious in* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that every best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in \mathcal{X}$. Action s^* is *contagious* if it is contagious in some unbounded local interaction system.

Note that contagion of s^* in $(\mathcal{X}, P)$ requires that, once the finite set Y of initial s^*-players is chosen, s^* be eventually played by all the players along *any* best response sequence.

One can define the notion of contagion differently. For example, one allows for simultaneous best responses, or requires only *some* best response sequence to converge. As we show in Appendix A.1, however, all such definitions turn out to be equivalent to the original one for any (generic) supermodular game. For the concreteness, we use the notion of contagion as defined in Definition 2 throughout the main text.

For uninvadability, the notion "almost all" is formalized by "except for a set of players whose weight with respect to P is finite".[5] For an action configuration σ and a subset of actions $S' \subset S$, we write

$$\sigma_P(S') = \frac{1}{2} \sum_{(x,y):\, \sigma(x)\in S' \text{ or } \sigma(y)\in S'} P(x, y).$$

In particular, for an action $s^* \in S$, $\sigma_P(S \setminus \{s^*\}) = (1/2)\sum_{(\sigma(x),\sigma(y))\neq(s^*,s^*)} P(x, y)$, which is the total weight of pairs of players who play action profiles other than (s^*, s^*).

Definition 3. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *uninvadable in* $(\mathcal{X}, P)$ if there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$. Action s^* is *uninvadable* if it is uninvadable in any unbounded local interaction system.

By definition, if s^* is contagious, then actions other than s^* are not uninvadable; if s^* is uninvadable, then actions other than s^* are not contagious.

[5] A finite set has a finite weight, and the converse is true if the neighborhood weights are bounded away from 0: i.e., for some $c > 0$, $\sum_y P(x, y) \geq c$ for all x.

Here, uninvadability as well as contagion are defined for the universal domain of unbounded networks. Our main result (Theorem 1) characterizes this strong (weak, resp.) form of uninvadability (contagion, resp.). In Section 5, we will consider several restricted domains of networks and examine whether an action that is invaded (contagious, resp.) in the universal domain becomes uninvadable (remains contagious, resp.) in restricted domains.

We conclude this section with the remark that the characterization of contagious actions would be immediate if we allowed for asymmetric interaction weights, $P(x, y) \neq P(y, x)$. That is, s^* is contagious in some network with asymmetric interaction weights if and only if (s^*, s^*) is a strict Nash equilibrium in u. Thus we focus on symmetric interaction weights hereafter.

3 The Bilingual Game

Hereafter, we consider the class of 3×3 games described in the Introduction. We denote the actions A, AB, and B by 0, 1, and 2, respectively, so that $S = \{0, 1, 2\}$, and let the payoff function $u \colon S \times S \to \mathbb{R}$ be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{3.1a}$$

where we assume

$$b < c \leq d < a, \quad a - c < d - b, \quad e > 0. \tag{3.1b}$$

Action profiles $(0, 0)$ and $(2, 2)$ are the only pure-strategy Nash equilibria. By the assumption that $d < a$, $(0, 0)$ Pareto-dominates $(2, 2)$, while by $a - c < d - b$, $(2, 2)$ pairwise risk-dominates $(0, 0)$.[6] By the additional assumption that $c \leq d$, this game is *supermodular* with respect to the order on actions $0 < 1 < 2$, i.e., $u(h', k) - u(h, k) \leq u(h', k') - u(h, k')$ if $h < h'$ and $k < k'$.

We will exploit the property of supermodular games, that the best response correspondence is nondecreasing in the stochastic dominance order. For $\pi, \pi' \in \Delta(S)$, we write $\pi \precsim \pi'$ (and $\pi' \succsim \pi$) if π' stochastically dominates π, i.e., if

$$\sum_{k \geq h} \pi_k \leq \sum_{k \geq h} \pi'_k$$

for all $h \in S$. If u is supermodular, then

$$\max br(\pi) \leq \max br(\pi')$$

[6] In Appendix A.8, we analyze the case where $(0, 0)$ is both Pareto-dominant and pairwise risk-dominant.

$$\min br(\pi) \leq \min br(\pi')$$

whenever $\pi \precsim \pi'$.

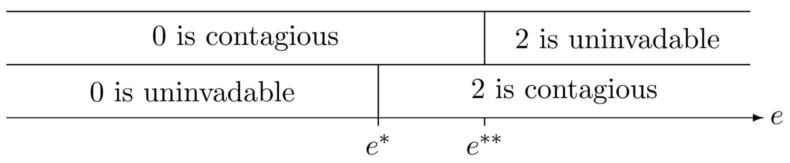
4 Characterization

In this section, we show that the Pareto-dominant action 0 prevails if the bilingual cost e is small, while the pairwise risk-dominant action 2 survives if e is large. The thresholds will be constructed based on two parameters:

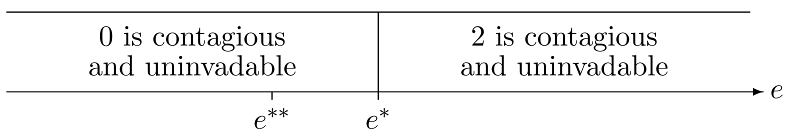
$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}.$$

Verify that $e^* \lesseqqgtr e^{**}$ if $c-b \lesseqqgtr a-c$. The following result characterizes contagious and uninvadable actions in the bilingual game, quantifying our argument in the Introduction.

Theorem 1. *Let u be the bilingual game given by* (3.1).
(i) 0 *is contagious if* $e < \max\{e^*, e^{**}\}$ *and uninvadable if* $e < e^*$. (ii) 2 *is contagious if* $e > e^*$ *and uninvadable if* $e > \max\{e^*, e^{**}\}$.



(1) $c-b < a-c$



(2) $c-b \geq a-c$

Note that for any (generic) value of e, at least one action is contagious and at most one action is uninvadable; when $e \in (e^*, e^{**})$ (which is nonempty if $c-b < a-c$), the two actions 0 and 2 are each contagious (in respective networks) and hence neither action is uninvadable.

One can verify that e^* and e^{**} increase as (i) b increases or c decreases or (ii) a and c increase by the same amount or d and b decrease by the same amount; that is, the contagion and uninvadability regions (in the space of e) of action 0 expand as action 0 becomes (i) less risky (i.e., b increases or c decreases) or (ii) more efficient (i.e., a increases with $a - c$ held fixed or d decreases with $d - b$ held fixed). This comparative statics is in stark contrast with that in the 2×2 case, where the risk-dominance and hence the characterizations for contagion and uninvadability are not affected by any payoff change with $a - c$ and $d - b$ held fixed.

In Subsections 4.1 and 4.2, we prove the contagion and the uninvadability parts of Theorem 1, respectively.

Example 1. Let $a = 11$, $b = 0$, $c = 3$, and $d = 10$: the game is represented by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 11 & 11 & 0 \\ 11-e & 11-e & 10-e \\ 3 & 10 & 10 \end{pmatrix} \end{array}.$$

Thus, $c - b = 3 < a - c = 8$, and $e^* = 5/3$ and $e^{**} = 40/19$. By Theorem 1, if $e > 40/19$, 2 is contagious and uninvadable; if $5/3 < e < 40/19$, both 0 and 2 are contagious; and if $e < 5/3$, 0 is contagious and uninvadable.

4.1 Contagion

We restate the contagion part of Theorem 1:

Proposition 1. *Let u be the bilingual game given by* (3.1)*.*
(i) 0 *is contagious if* $e < \max\{e^*, e^{**}\}$. (ii) 2 *is contagious if* $e > e^*$.

We decompose the proof into two lemmas. Lemma 1 provides sufficient conditions for contagion of actions 0 and 2 in general 3×3 supermodular games. Lemma 2 then checks by direct computation when those conditions are satisfied in the bilingual game. Our main theoretical contribution is in the proof of Lemma 1, where we explicitly construct networks in which contagion occurs as desired.

To better understand how contagion occurs in the bilingual game, consider a population of players indexed by integers $x \in \mathcal{X} = \mathbb{Z}$, where player x interacts with players $x \pm 1$ with equal weights; see Figure 1.

Figure 1: Nearest neighbor linear interaction

Suppose that at time $t = 0$, all players play B except for players -1, 0, and 1 who play A, and assume that the bilingual cost e is small so that $e < (a - d)/2$ (where $(a - d)/2 \leq e^*$). We demonstrate that A spreads contagiously. (For concreteness, we here consider a particular best response sequence, while one can verify that contagion occurs for all best response sequences as the definition requires.) Note that, since A is pairwise risk-dominated by B, no player is willing to switch from B to A. Suppose that player 2 adjusts his action at $t = 1$. With his two neighbors playing A and B, respectively, he abandons B and switches to AB since $e < (a - d)/2 \leq (a - c)/2$. Suppose next that player 3 revises his action at $t = 2$. Since he has one AB-neighbor and one B-neighbor, by $e < (a - d)/2$ he abandons B and switches to A or AB (depending on the payoff parameter values); let us assume that he chooses AB. Now let player 2 revise back again at $t = 3$. This time his neighbors are playing A and AB (instead of B), and hence he now switches to A. In this way, the region of A-players spreads, together with the "bilingual" region of AB-players between the A- and the B-regions; see Table 1.

	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$t = 0$	$\cdots$	B	A	A	A	B	B	B	$\cdots$
$t = 1$	$\cdots$	B	A	A	A	AB	B	B	$\cdots$
$t = 2$	$\cdots$	B	A	A	A	AB	AB	B	$\cdots$
$t = 3$	$\cdots$	B	A	A	A	A	AB	B	$\cdots$

Table 1: Contagion of action A

The above construction, which works only for $e < (a - d)/2$, is extended to obtain contagion of A for $e < e^*$ (and symmetrically that of B for $e > e^*$) in Lemma 1(1) where we construct a "linear" network with four neighbors (two for each side) with appropriately chosen weights (Figure 2). In order to obtain contagion further for the range $[e^*, e^{**})$ (which is nonempty when $c - b < a - c$), however, such a construction does not work and we need to construct a "non-linear" network in Lemma 1(2), in which different players may have different types of interacting neighborhoods (Figure 3).

For $p \in (0, 1/2)$ and $q, r \in (0, 1)$, $r \leq q$, let

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right),$$

and

$$\pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right),$$
$$\rho^c = \left(\tfrac{1-q}{2}, 0, \tfrac{1+q}{2}\right), \quad \rho^d = \left(\tfrac{1+r}{2}, 0, \tfrac{1-r}{2}\right), \quad \rho^e = \left(\tfrac{q-r}{2q}, \tfrac{q+r}{2q}, 0\right).$$

The conditions for contagion of actions 0 and 2 are stated in terms of best responses to the above mixed actions.

Lemma 1. *Let u be any 3×3 supermodular game.*

(1) (i) *If for some* $p \in (0, 1/2)$,

$$\max br(\pi^a) = 0, \quad \max br(\pi^b) \leq 1, \tag{4.1}$$

then 0 *is contagious.* (ii) *If for some* $p \in (0, 1/2)$,

$$\min br(\pi^a) \geq 1, \quad \min br(\pi^b) = 2, \tag{4.2}$$

then 2 *is contagious.*

(2) (i) *If for some* $q, r \in (0, 1)$ *with* $r \leq q$,

$$\max br(\pi^c) = 0, \quad \max br(\pi^d) \leq 1, \quad \max br(\pi^e) = 0, \tag{4.3}$$

then 0 *is contagious.* (ii) *If for some* $q, r \in (0, 1)$ *with* $r \leq q$,

$$\min br(\rho^c) = 2, \quad \min br(\rho^d) \geq 1, \quad \min br(\rho^e) = 2, \tag{4.4}$$

then 2 *is contagious.*

Proof. (1) Since cases (i) and (ii) are symmetric, we only show case (i). Let $p \in (0, 1/2)$ satisfy (4.1). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \mathbb{Z}$, and P be defined by

$$P(x, y) = \begin{cases} p & \text{if } |x - y| = 1 \\ \frac{1}{2} - p & \text{if } |x - y| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The defined local interaction system is depicted in Figure 2.

Figure 2: Linear interaction

We will use the following properties of this system.

Observation 1.

(a) If $\sigma(x-2) = \sigma(x-1) = 0$ and $\sigma(x+1) \leq 1$ (or symmetrically if $\sigma(x-1) \leq 1$ and $\sigma(x+1) = \sigma(x+2) = 0$), then $\max BR(\sigma|x) = 0$.

(b) If $\sigma(x-2) = 0$ and $\sigma(x-1) \leq 1$ (or symmetrically if $\sigma(x+1) \leq 1$ and $\sigma(x+2) = 0$), then $\max BR(\sigma|x) \leq 1$.

Proof. (a) Suppose that $\sigma(x-2) = \sigma(x-1) = 0$ and $\sigma(x+1) \leq 1$. Then by construction, the distribution over the actions of player x's neighbors, $\pi(\sigma|x) \in \Delta(S)$, satisfies

$$\pi(\sigma|x)(0) \geq P(x-2, x-1|x) = \frac{1}{2},$$
$$\pi(\sigma|x)(0) + \pi(\sigma|x)(1) \geq P(x-2, x-1, x+1|x) = \frac{1}{2} + p,$$

which implies that $\pi(\sigma|x) \precsim \pi^a = (1/2, p, 1/2 - p)$. By the assumption (4.1) and the supermodularity of u, it follows that $\max BR(\sigma|x) = 0$.

(b) Suppose that $\sigma(x-2) = 0$ and $\sigma(x-1) \leq 1$. Then by construction,

$$\pi(\sigma|x)(0) \geq P(x-2|x) = \frac{1}{2} - p,$$
$$\pi(\sigma|x)(0) + \pi(\sigma|x)(1) \geq P(x-2, x-1|x) = \frac{1}{2},$$

which implies that $\pi(\sigma|x) \precsim \pi^b = (1/2 - p, p, 1/2)$. By the assumption (4.1) and the supermodularity of u, it follows that $\max BR(\sigma|x) \leq 1$. ▮

Continuing the proof of Lemma 1(1), let $Y = \{-3, \ldots, 2\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t\to\infty} \sigma^t(x) = 0 \qquad (\diamondsuit_x)$$

holds for all $x \in \mathcal{X}$. We only consider players $x \geq 0$; the analogous argument applies to $x < 0$.

We first show $(\diamondsuit_0)$ and $(\diamondsuit_1)$, or more strongly, that

$$\sigma^t(x) = 0 \text{ for } x = -2, \ldots, 1$$
$$\sigma^t(x) \leq 1 \text{ for } x = -3, 2$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(x) = 2$ for $x = -2, \ldots, 1$ and $\sigma^t(x) \leq 1$ for $x = -3, 2$ by properties (a) and (b) in Observation 1, respectively.

Assume $(\diamondsuit_{x-2})$ and $(\diamondsuit_{x-1})$. Then, there exists T_0 such that $\sigma^t(x-2) = \sigma^t(x-1) = 0$ for all $t \geq T_0$. By Observation 1(b), this implies that there exists T_1 such that $\sigma^t(x) \leq 1$ for all $t \geq T_1$. By Observation 1(b) applied for $x+1$ in place of x, this implies that there exists T_2 such that $\sigma^t(x+1) \leq 1$ for all $t \geq T_2$. By Observation 1(a), this implies that there exists T_3 such that $\sigma^t(x) = 0$ for all $t \geq T_3$, meaning that $(\diamondsuit_x)$ holds.

(2) Since cases (i) and (ii) are symmetric, we only show case (i). Let $q, r \in (0, 1)$, $r \leq q$, satisfy (4.3). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$, and P be defined by

$$P((\alpha, i), (\alpha, j)) = \begin{cases} 1 - q & \text{if } |i - j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\alpha, i), (\beta, j)) = P((\beta, j), (\alpha, i)) = \begin{cases} q + r & \text{if } i = j \\ q - r & \text{if } i = j + 1 \text{ and } j \geq 0 \\ q - r & \text{if } i = j - 1 \text{ and } j \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\beta, i), (\beta, j)) = 0 \text{ for all } i, j.$$

The defined local interaction system is depicted in Figure 3.

Figure 3: Non-linear interaction

We will use the following properties of this system.

Observation 2.

(c) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = \sigma(\beta, i) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = \sigma(\beta, i) = 0$), then $\max BR(\sigma|(\alpha, i)) = 0$.

(d) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = 0$), then $\max BR(\sigma|(\alpha, i)) \leq 1$.

(e) If $\sigma(\alpha, i) \leq 1$, then $\max BR(\sigma|(\beta, i)) = 0$.

Proof. (c) Suppose that $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = \sigma(\beta, i) = 0$. Then by construction,

$$\pi(\sigma|(\alpha, i))(0) \geq P((\alpha, i-1), (\beta, i-1), (\beta, i)|(\alpha, i)) = \frac{1+q}{2},$$

which implies $\pi(\sigma|(\alpha, i)) \precsim \pi^c = ((1+q)/2, 0, (1-q)/2)$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\alpha, i)) = 0$.

(d) Suppose that $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = 0$. Then by construction,

$$\pi(\sigma|(\alpha, i))(0) \geq P((\alpha, i-1), (\beta, i-1)|(\alpha, i)) = \frac{1-r}{2},$$

which implies $\pi(\sigma|(\alpha,i)) \precsim \pi^d = ((1-r)/2, 0, (1+r)/2)$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\alpha,i)) \leq 1$.

(e) Suppose that $\sigma(\alpha,i) \leq 1$. Then by construction,

$$\pi(\sigma|(\beta,i))(0) + \pi(\sigma|(\beta,i))(1) \geq P((\alpha,i)|(\beta,i)) = \frac{q+r}{2q},$$

which implies $\pi(\sigma|(\beta,i)) \precsim \pi^e = (0, (q+r)/(2q), (q-r)/(2q))$. By the assumption (4.3) and the supermodularity of u, it follows that $\max BR(\sigma|(\beta,i)) = 0$. ▮

Continuing the proof of Lemma 1(2), let $Y = \{(\alpha,i) \mid i = -1,0,1\} \cup \{(\beta,i) \mid i = -1,0,1\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t\to\infty} \sigma^t(\alpha,i) = 0 \text{ and } \lim_{t\to\infty} \sigma^t(\beta,i) = 0 \qquad (\heartsuit_i)$$

holds for all $i \in \mathbb{Z}$. We only consider $i \geq 0$; the analogous argument applies to $i < 0$.

We first show $(\heartsuit_1)$, or more strongly, that

$$\sigma^t(\alpha,i) = \sigma^t(\beta,i) = 0 \text{ for } i = -1,0,1$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(\alpha,i) = \sigma^t(\beta,i) = 0$, $i = -1,0,1$, by properties (c) and (e) in Observation 2, respectively.

Assume $(\heartsuit_{i-1})$. Then, there exists T_0 such that $\sigma^t(\alpha,i-1) = \sigma^t(\beta,i-1) = 0$ for all $t \geq T_0$, By Observation 2(d), this implies that there exists T_1 such that $\sigma^t(\alpha,i) \leq 1$ for all $t \geq T_1$. By Observation 2(e), this implies that there exists T_2 such that $\sigma^t(\beta,i) = 0$ for all $t \geq T_2$. By Observation 2(c) this implies that there exists T_3 such that $\sigma^t(\alpha,i) = 0$ for all $t \geq T_3$. We thus obtain $(\heartsuit_i)$. ▮

Denote

$$e^\sharp = \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)}.$$

Verify that $e^{**} \lesseqgtr e^\sharp$ if $c-b \lesseqgtr a-c$. The following result characterizes when the hypotheses in Lemma 1 are satisfied in the bilingual game.

Lemma 2. *Let u be the bilingual game given by* (3.1).

(1) (i) *Condition* (4.1) *holds for some $p \in (0,1/2)$ if $e < e^*$.* (ii) *Condition* (4.2) *holds for some $p \in (0,1/2)$ if $e > e^*$.*

(2) *Condition* (4.3) *holds for some $0 < r \leq q < 1$ if $e < \min\{e^{**}, e^\sharp\}$.*

Proof. See Appendix A.2. ▮

Proof of Proposition 1. (i) Suppose that $e < \max\{e^*, e^{**}\}$. If $\max\{e^*, e^{**}\} = e^*$, then condition (4.1) holds for some $p \in (0, 1/2)$ by Lemma 2(1-i), and hence 0 is contagious by Lemma 1(1-i). If $\max\{e^*, e^{**}\} = e^{**}$, in which case $c - b < a - c$ and thus $\min\{e^{**}, e^\sharp\} = e^{**}$, then condition (4.3) holds for some $0 < r \leq q < 1$ by Lemma 2(2-i), and hence 0 is contagious by Lemma 1(2-i). In both cases, 0 is contagious.

(ii) Suppose that $e > e^*$. Then condition (4.2) holds for some $p \in (0, 1/2)$ by Lemma 2(1-ii), and hence 2 is contagious by Lemma 1(1-ii). ▮

Case (1) is consistent with the one-dimensional setup of Goyal and Janssen (1997). They show that under local interaction on a circle, 0 is contagious if $e < e^*$ while 2 is contagious if $e > e^*$.

4.2 Uninvadability

We restate the uninvadability part of Theorem 1:

Proposition 2. *Let u be the bilingual game given by* (3.1)*.*
(i) 0 *is uninvadable if* $e < e^*$*.* (ii) 2 *is uninvadable if* $e > \max\{e^*, e^{**}\}$*.*

The condition for uninvadability is stated by using the concept of *monotone potential maximizer* (*MP-maximizer*) due to Morris and Ui (2005). We employ its refinement, strict MP-maximizer, due to Oyama et al. (2008). For our purpose, we define it only for the smallest and the largest actions, which we denote by $\underline{s}$ and $\overline{s}$, respectively.[7] For a function $f\colon S \times S \to \mathbb{R}$ and a probability distribution $\pi \in \Delta(S)$, write $br_f(\pi) = \arg\max_{h \in S} f(h, \pi)$. (Thus the best response correspondence br for the game u as defined in (2.1) is now denoted br_u.) Function f is symmetric if $f(h, k) = f(k, h)$ for all $h, k \in S$ (i.e., it is a symmetric $|S| \times |S|$ matrix).

Definition 4. (i) $\underline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S \times S \to \mathbb{R}$ with $v(\underline{s}, \underline{s}) > v(h, k)$ for all $(h, k) \neq (\underline{s}, \underline{s})$ such that for all $\pi \in \Delta(S)$,

$$\max br_u(\pi) \leq \max br_v(\pi). \tag{4.5}$$

Such a function v is called a *strict MP-function* for $\underline{s}$.

(ii) $\overline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S \times S \to \mathbb{R}$ with $v(\overline{s}, \overline{s}) > v(h, k)$ for all $(h, k) \neq (\overline{s}, \overline{s})$ such that for all $\pi \in \Delta(S)$,

$$\min br_u(\pi) \geq \min br_v(\pi). \tag{4.6}$$

Such a function v is called a *strict MP-function* for $\overline{s}$.

A strict MP-maximizer is a strict Nash equilibrium and, in supermodular games, is unique if it exists (Oyama et al. (2008)).

[7] Here, we define for actions, rather than action profiles, since we only consider symmetric action profiles of symmetric games.

Lemma 3. *Let u be any game. If $s^* = \underline{s}, \overline{s}$ is a strict MP-maximizer of u with MP-function v and if u or v is supermodular, then s^* is uninvadable.*

Proof. See Appendix A.3. ▮

In the bilingual game, Proposition 1 and Lemma 3 imply that action 0 (2, resp.) is never a strict MP-maximizer if $e > e^*$ ($e < \max\{e^*, e^{**}\}$, resp.), and hence, no strict MP-maximizer exists if $e^* < e < \max\{e^*, e^{**}\}$. The following lemma establishes existence of a strict MP-maximizer for the remaining cases (except for knife-edge values of e).

Lemma 4. *Let u be the bilingual game given by* (3.1)*.*

(i) 0 *is a strict MP-maximizer if* $e < e^*$. (ii) 2 *is a strict MP-maximizer if* $e > \max\{e^*, e^{**}\}$.

Proof. See Appendix A.4. ▮

In 2×2 coordination games, a risk-dominant equilibrium is a strict MP-maximizer. Beyond 2×2 games, no general method to find an MP-maximizer has been known (except for some special cases). A strict MP-maximizer is shown, by ad hoc construction, to generically exist in symmetric 3×3 supermodular games such that the three symmetric action profiles are all Nash equilibria (Morris (1999), Oyama and Takahashi (2009)), whereas it fails to exist in some 3×3 games with two strict Nash equilibria, e.g., in our bilingual game with $e^* < e < e^{**}$ (see also Honda (2011)). The proof of Lemma 4 is here again by ad hoc construction of an MP-function involving tedious computations.

5 (Non-)Critical Classes of Networks

Thus far, we have allowed for the universal domain of all networks, and in particular, for our contagion result (Lemma 1) we had a maximal freedom to choose a network to obtain contagion. There, the constructed networks have different contagious actions for the same set of payoff parameters. This, in turn, suggests that one can differentiate (subclasses of) networks by analyzing strategic behavior on each network for various payoff parameters of our bilingual game. In this section, we study several subclasses of networks and derive conditions for contagion and uninvadability in these subclasses. This exercise enables us to classify networks in terms of those conditions in the bilingual game, which will provide a finer analysis than the one based on 2×2 coordination games.

In particular, we examine whether a given class of networks is critical for contagion. Formally, for a given game u and for a class $\mathcal{C}$ of unbounded networks, action s^* is *contagious in* $\mathcal{C}$ (*uninvadable in* $\mathcal{C}$, resp.) if it is contagious in some network in $\mathcal{C}$ (uninvadable in every network in $\mathcal{C}$, resp.). We say that a class $\mathcal{C}$ is *critical for contagion* if any action s^* that is contagious in the universal domain is also contagious in $\mathcal{C}$. In that case, one can restrict attention to that class to characterize contagious actions. Conversely, if $\mathcal{C}$ is non-critical for contagion, some action is contagious in no network in $\mathcal{C}$ but in some network outside $\mathcal{C}$. For example, if the game u is a 2×2 coordination game, a risk-dominant equilibrium is contagious in the network in Figure 1, and hence that network forms a (singleton) critical class for contagion. On the other hand, if u is the bilingual game, it follows from our analysis in the previous section that the network in Figure 1 is not critical for some parameter values, while the union of two classes of networks given by Figures 2 and 3 is critical.

In what follows, we consider two classes of "simple" networks, which we call linear and multidimensional lattice networks, and show that these classes of networks are not critical for contagion in the bilingual game.

5.1 Linear Networks

We first introduce linear networks and analyze contagion and uninvadability in those networks. A network $(\mathcal{X}, P)$ is *linear* if $\mathcal{X} = \mathbb{Z}$ and interaction weights P are invariant up to translation: $P(x, y) = P(x + z, y + z)$ for any $x, y, z \in \mathbb{Z}$. (Note that any linear network is unbounded.) Clearly, both of the networks in Figure 1 and in Figure 2 are linear. On the other hand, the network in Figure 3 is not linear.[8]

Due to the translation invariance and symmetry of P, for each $y \in \mathbb{Z}$ we have $P(0, y) = P(-y, 0) = P(0, -y)$, hence $P(y|0) = P(-y|0)$. Conversely, conditional weights $P(y|0)$ of player 0 determine translation invariant weights $P(x, y)$ uniquely (up to positive constant multiplication) if $P(0|0) = 0$, and $P(y|0)$ satisfies reflection symmetry, i.e., $P(y|0) = P(-y|0)$ for all $y > 0$.

It follows from the proof of Lemma 1 that in the class of linear networks given in Figure 2, action 0 (2, resp.) is contagious if $e < e^*$ ($e > e^*$, resp.). The following theorem shows that these conditions are also sufficient for uninvadability in the class of all linear networks.

Theorem 2. *Let u be the bilingual game given by* (3.1)*.*

(i) 0 *is contagious and uninvadable in the class of linear networks if* $e < e^*$*.* (ii) 2 *is contagious and uninvadable in the class of linear networks if* $e > e^*$*.*

Proof. See Appendix A.5. ▮

The characterization given in this theorem differs from the one for the universal domain given in Theorem 1 when $c - b < a - c$ and $e^* < e < e^{**}$,

[8] Even if we map $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$ to $\mathbb{Z}$ by relabeling (α, i) with $2i$ and (β, i) with $2i+1$, interaction weights do not satisfy translation invariance.

which implies that in this range of parameter values, the class of linear networks is *not* critical for contagion.

This characterization generalizes to a slightly larger class of networks where each node on a line is replicated into finitely many nodes. Formally, $(\mathcal{X}, P)$ is a *replicated linear network* if $\mathcal{X} = \{1, \ldots, m\} \times \mathbb{Z}$ and P is invariant up to translation, i.e., $P(x, y) = P(x + z, y + z)$ for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \{1, \ldots, m\} \times \mathbb{Z}$, where sums in the first coordinate, $x_1 + z_1$ and $y_1 + z_1$, are defined modulo m. For example, the network depicted in Figure 4 is a replicated linear network with $m = 3$,[9] whereas the network in Figure 3 is not.

Figure 4: Replicated linear network

Theorem 3. *Let u be the bilingual game given by* (3.1)*.*

(i) 0 *is contagious and uninvadable in the class of replicated linear networks if* $e < e^*$*.* (ii) 2 *is contagious and uninvadable in the class of replicated linear networks if* $e > e^*$*.*

The proof is analogous to that of Theorem 2 and thus omitted. This theorem implies that the class of all replicated linear networks is not critical for contagion.

5.2 Multidimensional Lattice Networks

We next show that the characterization in the previous subsection generalizes to multidimensional lattice networks with translation invariant interaction weights. For the sake of concreteness, we here focus on the m-dimensional lattice with *n-max distance interactions*, where each player interacts with all players within n steps away in each of the m coordinates, i.e., $\mathcal{X} = \mathbb{Z}^m$, and $P(x, y) = 1$ if $1 \leq \max_{i=1,\ldots,m} |x_i - y_i| \leq n$ and $P(x, y) = 0$ otherwise. A more general treatment is relegated to Appendix A.6, where we consider a broader class of networks on $\mathbb{Z}^m$ such that interaction weights $P(x, y)$ are translation invariant and conditional weights $P(x|0)$ are approximated (with an appropriate normalization) by a density function on $\mathbb{R}^m$.

[9] The "thick line graph" in Immorlica et al. (2007, Figure 2) is another example.

For 2×2 coordination games, Morris (2000) demonstrates that the characterization for contagion and uninvadability in the linear lattice still holds with higher dimensions as long as the interaction radius n is sufficiently large. We obtain an analogous characterization for our bilingual game.[10]

Theorem 4. *Let u be the bilingual game given by* (3.1)*. Fix the dimension m.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 0 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 2 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.*

Proof. See Appendix A.6. ▮

The proof is analogous to that of Lemma 1. In the case of $e < e^*$, for example, we show the contagion of action 0 by an induction argument along a sequence of regions of 0-players surrounded by "bilingual" regions. Here, each 0-player region is the set of lattice (i.e., integer-coordinate) points contained in a large m-dimensional ball with an outer m-dimensional ring of 1-players.

To conclude, the class of n-max distance interaction networks (with large n) as well as the class of (replicated) linear networks are not critical for contagion, and hence the network in Figure 3 exhibits fundamentally different properties in strategic behavior from those simple networks.

6 Comparison of Networks

Extending the analysis in Section 5, one could ask whether a given action is contagious or not in each network for each payoff-parameter value of the bilingual game. This question, however, seems to be intractable or have no insightful answer unless we restrict attention to well-structured networks as in Section 5. Instead, this section asks a comparative question. For a pair of networks, which network is more likely to induce contagion? Or more precisely, in which network is the set of payoff-parameter values for which a given action is contagious larger (with respect to the set-inclusion order)? It turns out that such comparison induces a non-trivial preorder (a transitive and reflexive binary relation) that reflects underlying network structures.

For concreteness, we use the class of bilingual games as "test functions" to measure the relative power of inducing contagion, but Theorem 5 below actually extends to all symmetric supermodular games.

Definition 5. A network $(\mathcal{X}, P)$ is *more contagion-inducing* than another network $(\hat{\mathcal{X}}, \hat{P})$ if for any payoff-parameter value of the bilingual game, an action s^* is contagious in $(\mathcal{X}, P)$ whenever s^* is contagious in $(\hat{\mathcal{X}}, \hat{P})$.

[10] Even if we allow for small n, the class of max distance interaction networks is not critical for contagion. See Example 7 in Section 6.

This notion induces a preorder over the class of networks. As the following example shows, this preorder is not complete.

Example 2 (Figure 2 versus Figure 3)**.** Fix a bilingual game that satisfies $e \in (e^*, e^{**})$. Recall the two networks in the proof of Lemma 1, where we choose p, q, and r that satisfy (4.2) and (4.3). Then B is contagious in the network of Figure 2, whereas A is contagious in the network of Figure 3. Thus neither network is more contagion-inducing than the other.

Incompleteness is in contrast to the analysis of contagion for 2×2 coordination games. For 2×2 coordination games, all networks are ranked in a complete preorder (i.e., a "preference" order) according to the "q-cohesiveness" of these networks, and in particular, the network in Figure 1 is the most contagion-inducing (Morris (2000)).

The next example shows that a network may be strictly more contagion-inducing than another network.

Example 3 (Tree versus ladder)**.** Consider the "tree" depicted in Figure 5, where each player is indexed by a finite sequence of 0 or 1, $\mathcal{X} = \bigcup_{n \geq 0} \{0, 1\}^n$, and player $x \in \mathcal{X}$ interacts with $x0$, $x1$, and x^- with equal weights, where x^- is the immediate predecessor of x, i.e., the truncation of x that removes the last digit of x. Also consider the "ladder" depicted in Figure 6, where each player is indexed by a pair of α or β and an integer, $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$, and with equal weights, player (α, i) interacts with $(\alpha, i \pm 1)$ and (β, i), and player (β, i) interacts with (α, i) and $(\beta, i \pm 1)$. Then we can show that the ladder is more contagion-inducing than the tree. We postpone the proof of this statement until Example 4 below. Here we verify it in the following two scenarios. A first scenario is that B spreads contagiously from the root of the tree by inducing players 0 and 1 to switch from A to B, then players 00, 01, 10, and 11 to switch from A to B, and so on. Then B is the unique best response against the mixture of actions A and B with probabilities 2/3 and 1/3, so B spreads contagiously from any node in the ladder. In another scenario, B also spreads from the root of the tree, but in the following way: players 0 and 1 first switch from A to AB, then players 00, 01, 10, and 11 switch from A to AB, and then players 0 and 1 switch from AB to B. In this scenario, B spreads contagiously in the ladder as follows: B at $(\alpha, 0)$ induces AB at $(\alpha, \pm 1)$ and $(\beta, 0)$, which further induces AB or B at $(\alpha, \pm 2)$ and $(\beta, \pm 1)$, and then B at $(\alpha, \pm 1)$ and $(\beta, 0)$.

Moreover, the ladder is *strictly* more contagion-inducing than the tree. To see this, consider a bilingual game that satisfies

$$br(2/3, 1/3, 0) = \{A\},$$
$$br(2/3, 0, 1/3) = \{AB\},$$
$$br(1/3, 1/3, 1/3) = \{B\},$$

Figure 5: Tree

Figure 6: Ladder

that is,

$$2a - c < 2d - b, \qquad \frac{2a - c - d}{3} < e < \frac{2(a - c)}{3},$$

which is satisfied, for example, by $(a, b, c, d, e) = (6, 0, 3, 5, 2)$. For this game, action B is contagious in the ladder, but not in the tree. To see this, for any finite set Y of initial B-players in the tree, pick a maximal (longest) element x of Y and assume that all successors of x play A. Then players $x0$ and $x1$ may switch from A to AB, but all the other successors of x continue to play A in any best response sequence. On the other hand, in the ladder, if players $(\alpha, 0)$, $(\alpha, 1)$, $(\beta, 0)$, and $(\beta, 1)$ play B, then players $(\alpha, 2)$ and $(\beta, 2)$ switch from A to AB, and induce each other to switch further from AB to B. By a similar argument, all players subsequently switch from A to AB, and to B.

Note that if we restrict attention to 2×2 coordination games, action B is contagious in the tree or in the ladder if and only if B is a $1/3$-dominant equilibrium, i.e., $2a + b < 2c + d$, so the tree is as contagion-inducing as the ladder. Thus, we cannot distinguish the tree from the ladder in terms of contagion for 2×2 coordination games, but we can rank them with a strict order if we use the bilingual game. This is reminiscent of the discussion in

Section 5, which suggested that the bilingual game reveals finer structures of networks than those that 2×2 coordination games can reveal.

In Example 3, notice that the tree looks more and more similar to the ladder as we "bundle" together 01 and 10; 001, 010, and 100; 0 and 101; 1 and 011, etc. This similarity is why any contagion in the tree can be replicated in the ladder. The following notion formalizes the idea of "bundling".

Definition 6. For two networks $(\mathcal{X}, P)$ and $(\hat{\mathcal{X}}, \hat{P})$, a mapping $\varphi\colon \mathcal{X} \to \hat{\mathcal{X}}$ is a *weight-preserving node identification* if φ is onto and there exists a finite subset E of $\mathcal{X}$ such that for any $x \in \mathcal{X} \setminus E$ and any $\hat{y} \in \hat{\mathcal{X}}$,

$$\hat{P}(\varphi(x), \hat{y}) = \sum_{y \in \varphi^{-1}(\hat{y})} P(x, y).$$

The weight-preserving property implies an analogous property on conditional weights as follows:

$$\begin{aligned} \hat{P}(\hat{y}|\varphi(x)) &= \frac{\hat{P}(\varphi(x), \hat{y})}{\sum_{\hat{z} \in \hat{\mathcal{X}}} \hat{P}(\varphi(x), \hat{z})} \\ &= \sum_{y \in \varphi^{-1}(\hat{y})} \frac{P(x, y)}{\sum_{\hat{z} \in \hat{\mathcal{X}}} \sum_{z \in \varphi^{-1}(\hat{z})} P(x, z)} \\ &= \sum_{y \in \varphi^{-1}(\hat{y})} \frac{P(x, y)}{\sum_{z \in \mathcal{X}} P(x, z)} = \sum_{y \in \varphi^{-1}(\hat{y})} P(y|x) \end{aligned}$$

for any $x \in \mathcal{X} \setminus E$ and any $\hat{y} \in \hat{\mathcal{X}}$. A node in E is called an *exceptional node.*[11]

Theorem 5. *If there exists a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$, then $(\hat{\mathcal{X}}, \hat{P})$ is more contagion-inducing than $(\mathcal{X}, P)$.*

Proof. See Appendix A.7. ▌

The main idea of the proof is as follows. Suppose that s^* is contagious in $(\mathcal{X}, P)$, and φ is a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$. Take any best response sequence $(\hat{\sigma}^t)$ on $(\hat{\mathcal{X}}, \hat{P})$. We construct another sequence (σ^t) on $(\mathcal{X}, P)$ by $\sigma^t(x) = \hat{\sigma}^t(\varphi(x))$ for any $x \in \mathcal{X}$ and $t \geq 0$, which, by the definition of weight-preserving node identification, is "almost" a best response sequence. Since s^* is contagious in $(\mathcal{X}, P)$, $(\sigma^t(x))$ converges to s^* for any $x \in \mathcal{X}$, and hence $(\hat{\sigma}^t(\hat{x}))$ also converges to s^* for any $\hat{x} \in \hat{\mathcal{X}}$. This argument, however, has two issues: along the sequence (σ^t), players in each $\varphi^{-1}(\hat{x})$ change actions simultaneously, which violates

[11] Allowing for exceptional nodes is essential for constructing a weight-preserving node identification from the tree to the ladder (see Example 4), but the reader may want to assume $E = \emptyset$ on the first reading.

property (i) in Definition 1, and players at exceptional nodes may not play best responses. We use Lemma A.1 in Appendix A.1 to resolve those issues.

In the next four examples, we construct weight-preserving node identifications and illustrate the implications of Theorem 5.

Example 4 (Tree versus ladder, continued)**.** There exists a weight-preserving node identification from the tree to the ladder. In fact, one can construct infinitely many such mappings recursively as follows: given that each node $x \in \bigcup_{m \geq 0}^{n} \{0,1\}^m$ of the tree with depth at most n is assigned with a node $\varphi(x)$ of the ladder, for each $x \in \{0,1\}^n$, we assign $x0$ and $x1$ with two of the neighbors of $\varphi(x)$ in the ladder other than $\varphi(x^-)$. We can always do so since each node on the ladder has three neighbors. For example,

$$\begin{aligned} \varphi(\emptyset) &= (\alpha, 0), \\ \varphi(0) &= (\alpha, 1), \varphi(1) = (\beta, 0), \\ \varphi(00) &= (\alpha, 2), \varphi(01) = \varphi(10) = (\beta, 1), \varphi(11) = (\beta, -1), \ldots \end{aligned}$$

Then φ preserves interaction weights except at the root $\emptyset$. (φ does not preserve interaction weights at the root $\emptyset$ because player $\emptyset$ has two neighbors in the tree while player $\varphi(\emptyset)$ has three neighbors in the ladder.) Thus, by Theorem 5, the tree is more contagion-inducing than the ladder.

Example 5 (Line versus lattice)**.** Consider the line depicted in Figure 1 and the two-dimensional lattice depicted in Figure 7, where each player $x = (x_1, x_2) \in \mathbb{Z}^2$ interacts with $(x_1 \pm 1, x_2)$ and $(x_1, x_2 \pm 1)$ with equal weights. Then the mapping $\varphi(x_1, x_2) = x_1 + x_2$ is a weight-preserving node identification from the two-dimensional lattice to the line with no exceptional node. Thus, by Theorem 5, the line is more contagion-inducing (in fact, strictly more contagion-inducing) than the two-dimensional lattice.

Morris (2000) showed that the line is more contagion-inducing than the two-dimensional lattice for the class of 2×2 coordination games by computing "contagion thresholds" explicitly. Our Theorem 5 gives an alternative proof to this result, which generalizes to other pairs of networks and to the bilingual game (in fact, to all symmetric supermodular games).

Example 6 (Replicated lines versus lattice)**.** Consider the replicated linear network in Figure 4 (with equal weights) and the two-dimensional lattice in Figure 7. Then $\varphi(x_1, x_2) = (x_1 - 3\lceil x_1/3 \rceil + 1, x_2)$ is a weight-preserving node identification from the two-dimensional lattice to the replicated linear network with no exceptional node. Thus, by Theorem 5, the replicated line is more contagion-inducing (in fact, strictly more contagion-inducing) than the two-dimensional lattice.

Example 7 (Line versus max distance)**.** Consider the n-max distance interaction network on the m-dimensional lattice $\mathbb{Z}^m$. Define the mapping

Figure 7: Two-dimensional lattice

$\varphi\colon \mathbb{Z}^m \to \mathbb{Z}$ by

$$\varphi(x_1, \ldots, x_m) = x_1 + (n+1)x_2 + \cdots + (n+1)^{m-1}x_m$$

for any $(x_1, \ldots, x_m) \in \mathbb{Z}^m$. Since φ is linear and maps any pair of neighboring points in $\mathbb{Z}^m$ to different points in $\mathbb{Z}$, φ is a weight-preserving node identification (with no exceptional node) from the n-max distance interaction network on $\mathbb{Z}^m$ to a linear network on $\mathbb{Z}$ with weights $P_{m,n}(x, y) = \#(\varphi^{-1}(y-x) \cap [-n, n]^m)$ for any $x, y \in \mathbb{Z}$ with $x \neq y$.[12] Thus, by Theorem 5, the linear network $(\mathbb{Z}, P_{m,n})$ is more contagion-inducing than the n-max distance interaction network on $\mathbb{Z}^m$. Combined with the non-criticality of the class of all linear networks (Theorem 2), this implies that the class of all max distance interaction networks is not critical for contagion.

As the next two examples illustrate, there may not exist any weight-preserving node identification.

Example 8 (Line versus Figure 3)**.** The mapping $\varphi(\alpha, i) = \varphi(\beta, i) = i$ from the network in Figure 3 to $\mathbb{Z}$ does not preserve interaction weights since (α, i) and (β, i) have different neighborhood structures. In fact, Theorems 2 and 5 imply that no node identification from the network in Figure 3 to any linear network preserves interaction weights.

Example 9 (Line versus line)**.** Consider two linear networks, one depicted in Figure 1 and the other in Figure 2 with $p \in (0, 1/2)$. Then it is not difficult to see that Figure 2 is strictly more contagion-inducing than Figure 1. However, no node identification from Figure 1 to Figure 2 preserves interaction weights

[12] $\#X$ denotes the cardinality of X.

because any weight-preserving node identification can increase the number of neighbors only for exceptional nodes. Thus the converse of Theorem 5 is false, that is, the existence of weight-preserving node identifications is not necessary for comparing two networks.

7 Interpretations in Incomplete Information Games

Local interaction games and incomplete information games, though capturing different economic or social situations, share the same formal structures and thus belong to a more general class of "interaction games" (Morris (1997, 1999), Morris and Shin (2003)): in local interaction games, each node interacts with a set of neighbors and payoffs are given by the weighted sum of those from the interactions; in incomplete information games, each type interacts with a subset of types and payoffs are given by the expectation of those from the interactions. Indeed, Morris (1997, 1999) demonstrates, in spite of some technical differences, that several tools and results in the context of incomplete information games can be utilized also in the context of local interaction games, and vice versa.[13] In this section, we interpret our results, in particular the discussions in the previous section, in the language of incomplete information games, thereby shedding new lights on two existing lines of literature, robustness to incomplete information (Kajii and Morris (1997), Morris and Ui (2005)) and global games (Carlsson and van Damme (1993), Frankel et al. (2003)).

7.1 Robustness to Incomplete Information

A Nash equilibrium (s_1^*, s_2^*) of a two-player game u is said to be *robust to incomplete information* if every incomplete information game in which the payoffs are given by u with high probability has a Bayesian Nash equilibrium that plays (s_1^*, s_2^*) with high probability (Kajii and Morris (1997)). Robustness to incomplete information corresponds to uninvadability in local interaction systems in that both notions require that a small amount of "crazy types" should not affect the aggregate behavior.

Indeed, they have the same characterizations in many classes of games, including games with an MP-maximizer. In parallel with Lemma 3, an MP-maximizer of a game u with MP-function v is robust to incomplete information if u or v is supermodular (Morris and Ui (2005)). Combining

[13] A class of dynamic games with Poisson action revisions due to Matsui and Matsuyama (1995) (perfect foresight dynamics) also belong to interaction games, where each revising player interacts with a set of past and future players and payoffs are given the discounted sum of flow payoffs from the interactions (Takahashi (2008)).

this result with Lemma 4, we obtain a sufficient condition for robustness in the bilingual game.

Conversely, a necessary condition for robustness is obtained by constructing incomplete information games in which a given action profile is never played in equilibrium. Specifically, in any 3×3 supermodular game u, adjusting the proof of Lemma 1, one can construct incomplete information games in which the payoffs are given by u with probability arbitrarily close to 1 and playing 0 (2, resp.) everywhere is a unique rationalizable strategy if (4.1) ((4.2), resp.) holds for some $p \in (0, 1/2)$, or (4.3) ((4.4), resp.) holds for some $q, r \in (0, 1)$ with $r \leq q$ (Oyama and Takahashi (2011)). The necessary condition thus follows by applying this result to the bilingual game combined with Lemma 2.

These arguments characterize exactly as in Theorem 1 when an equilibrium in the bilingual game is robust to incomplete information.

Proposition 3. *Let u be the bilingual game given by* (3.1).

(i) $(0, 0)$ *is a unique robust equilibrium if* $e < e^*$. (ii) $(2, 2)$ *is a unique robust equilibrium if* $e > \max\{e^*, e^{**}\}$. (iii) *No action profile is robust if* $e^* < e < \max\{e^*, e^{**}\}$.

7.2 Global Games

Global games constitute a subclass of incomplete information games, where the underlying state θ is drawn from the real line, and each player i receives a noisy signal $x_i = \theta + \nu\varepsilon_i$ with ε_i being a noise error independent across players and from θ. Under supermodularity and state-monotonicity in payoffs, it is shown by a contagion argument that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$, while the limit equilibrium may depend on the distribution of noise terms ε_i (Frankel et al. (2003)).

Global game perturbations in the class of all incomplete information perturbations can be viewed as linear networks in the class of all networks. In global games, the distribution of the opponent's signal x_j conditional on x_i is (approximately) invariant up to translation (for small $\nu > 0$) due to the assumption of state-independent noise errors, which parallels the translation invariance in linear networks. In fact, one can mimic the argument by Frankel et al. (2003) and show that in a generic supermodular game, there exists at least one contagious action, and hence if an action is uninvadable, then it is also contagious and no other action is uninvadable.[14] Also, Basteck and Daniëls (2011) prove that, in any global game of 3×3 supermodular games independently of the noise distribution, action profile $(0, 0)$ ($(2, 2)$, resp.) is played at θ as $\nu \to 0$ if (4.1) ((4.2), resp.) holds for some $p \in (0, 1/2)$ at that state θ. Together with Lemma 2(1), this leads to the following

[14] In the special case of the bilingual game, these results follow from Theorem 1.

characterization of global-game noise-independent selection in the bilingual game, the same one as in Theorem 2.

Proposition 4. *Let u be the bilingual game given by* (3.1)*.*

(i) $(0,0)$ *is a noise-independent global game selection if* $e < e^*$*.* (ii) $(2,2)$ *is a noise-independent global game selection if* $e > e^*$*.*

Since this characterization is different from that in Proposition 3, global games are not a critical class of incomplete information games that determines whether or not an action profile is robust to incomplete information. See Oyama and Takahashi (2011) for further discussions.

8 Conclusion

We investigated contagion and uninvadability of actions for the bilingual game, and revealed finer structures of networks than what the analysis of 2×2 coordination games can distinguish. Of course, there are many questions that we did not address. For example, how can we characterize contagion and uninvadability for general (supermodular) games? How can we characterize the preorder between two networks if there is no weight-preserving node identification? Can we apply the notion of weight-preserving node identifications to other contexts? These questions remain for future research.

Appendix

A.1 Equivalent Definitions of Contagion in Supermodular Games

In this appendix, we discuss three other definitions of contagion, and show that all of them are equivalent to the original one for any (generic) symmetric supermodular game (S, u), where $S = \{0, 1, \ldots, n\}$ and $u(h', k) - u(h, k) \leq u(h', k') - u(h, k')$ if $h < h'$ and $k < k'$. (None of the results here relies on the particular payoff structure of the bilingual game.) We use the partial order $\sigma \leq \sigma'$ whenever $\sigma(x) \leq \sigma'(x)$ for any $x \in \mathcal{X}$.

First, recall that in the main text, we consider the sequential best response dynamics, where at most one player revises his action in each period (property (i) in Definition 1). Instead, we can define the simultaneous (resp. generalized) best response dynamics, where all (resp. some) players revise their actions at a time.

Definition A.1. A sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *simultaneous best response sequence* if $\sigma^t(x) \in BR(\sigma^{t-1}|x)$ for all $x \in \mathcal{X}$ and $t \geq 1$. A sequence $(\sigma^t)_{t=0}^{\infty}$ is a *generalized best response sequence* if it satisfies the following properties: (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and

(iii) if there exists $T \geq 0$ such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then $\lim_{t\to\infty} \sigma^t(x) \neq s$.

For clarity, we add adjective "sequential" to the original notion of best response sequences. Generalized best response sequences subsume both sequential and simultaneous best response sequences as special cases.

Using simultaneous or generalized best response sequences, we obtain two new definitions of contagion.[15]

Definition A.2. Action s^* is *contagious by simultaneous* (resp. *generalized*) *best responses in network* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that every simultaneous (resp. generalized) best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in \mathcal{X}$.

We refer to the notion of contagion in Definition 2 as "contagion by sequential best responses". By definition, contagion by generalized best responses implies both contagion by sequential best responses and by simultaneous best responses. Here we show the converse.

In the next lemma, we show that if s^* is contagious by sequential best responses, then there exist two sequential best response sequences that converge to s^* monotonically (one increasingly and the other decreasingly), and that any generalized best response sequence that starts between the two sequences also converges to s^*. This lemma is used to prove both Proposition A.1 below and Theorem 5 in the main text.

Lemma A.1. *Fix a network* $(\mathcal{X}, P)$ *and a supermodular game* u*. Suppose that* s^* *is contagious by sequential best responses in* $(\mathcal{X}, P)$*. Then there exist two sequential best response sequences* $(\sigma_-^t)_{t=0}^{\infty}$ *and* $(\sigma_+^t)_{t=0}^{\infty}$ *such that*

(1) $\sigma_-^t(x) \leq s^* \leq \sigma_+^t(x)$ *for all* $x \in \mathcal{X}$ *and* $t \geq 0$*;*

(2) $\sigma_-^0(x) = 0$ *and* $\sigma_+^0(x) = n$ *for all but finitely many* $x \in \mathcal{X}$*;*

(3) $\sigma_-^t(x) \in \{\sigma_-^{t-1}(x), \min BR(\sigma_-^{t-1}|x)\}$ *for all* $x \in \mathcal{X}$ *and* $t \geq 1$*;*

(4) $\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$ *for all* $x \in \mathcal{X}$*; and*

(5) $\min BR(\sigma_-^0|x) \geq \sigma_-^0(x)$ *and* $\max BR(\sigma_+^0|x) \leq \sigma_+^0(x)$ *for all* $x \in \mathcal{X}$*.*

Moreover,

(6) *for any generalized best response sequence* $(\tilde{\sigma}^t)_{t=0}^{\infty}$ *with* $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$*, we have* $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ *for all* $x \in \mathcal{X}$*.*

[15] The notion of contagion used in Morris (2000) is similar to contagion by simultaneous best responses, but requires only that for each $x \in \mathcal{X}$, $\sigma^t(x) = s^*$ for some $t \geq 0$.

Proof. Suppose that s^* is contagious by sequential best responses in $(\mathcal{X}, P)$ (and hence a strict Nash equilibrium). Let $Y \subset \mathcal{X}$ be a finite set as in Definition 2, and let $(\phi_-^t)_{t=0}^\infty$ be the sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = 0$ for all $x \in \mathcal{X} \setminus Y$, and $\phi_-^t(x) \in \{\phi_-^{t-1}(x), \min BR(\phi_-^{t-1}|x)\}$ for all $x \in \mathcal{X}$ and $t \geq 1$. By definition, $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in \mathcal{X}$.

The sequence $(\phi_-^t)_{t=0}^\infty$ satisfies properties (1)–(4), but not necessarily property (5). From $(\phi_-^t)_{t=0}^\infty$, we construct another sequence that satisfies property (5) as well. Let $\psi_-^0 = \phi_-^0$ and

$$\psi_-^t(x) = \begin{cases} \psi_-^{t-1}(x) & \text{if } \phi_-^t(x) \leq \psi_-^{t-1}(x), \\ \min BR(\psi_-^{t-1}|x) & \text{if } \phi_-^t(x) > \psi_-^{t-1}(x). \end{cases} \tag{A.1}$$

Clearly, $(\psi_-^t)_{t=0}^\infty$ is a sequential best response sequence. By the construction of $(\phi_-^t)_{t=0}^\infty$ and $(\psi_-^t)_{t=0}^\infty$ along with the supermodularity and s^* being a Nash equilibrium, one can show by induction on t that $\phi_-^t(x) \leq \psi_-^t(x) \leq s^*$ for all $x \in \mathcal{X}$ and $t \geq 0$. Thus for each $x \in \mathcal{X}$, $(\psi_-^t(x))_{t=0}^\infty$ is weakly increasing and converges to s^*.

Since s^* is a strict Nash equilibrium, we can take a finite but sufficiently large subset Z of $\bigcup_{x\in Y} \Gamma(x)$ such that for any $x \in Y$, the best response of player x is s^* if all players in Z play s^* (recall that $\Gamma(x)$ is the set of the neighbors of player x). Let T be sufficiently large so that $\psi_-^T(x) = s^*$ for all $x \in Z$.

We claim that $\min BR(\psi_-^T|x) \geq \psi_-^T(x)$ for all $x \in \mathcal{X}$. For $x \in Y$, since all players in Z play s^* in period T, we have $\min BR(\psi_-^T|x) = s^* \geq \psi_-^T(x)$. For $x \in \mathcal{X} \setminus Y$, we first have $\min BR(\psi_-^0|x) \geq 0 = \psi_-^0(x)$. Next, assume that $\min BR(\psi_-^{t-1}|x) \geq \psi_-^{t-1}(x)$. By the construction of $(\psi_-^t(x))_{t=0}^\infty$ in (A.1), $\psi_-^t(x)$ is equal to either $\psi_-^{t-1}(x)$ or $\min BR(\psi_-^{t-1}|x)$. In both cases, we have $\min BR(\psi_-^{t-1}|x) \geq \psi_-^t(x)$. Since $(\psi_-^t)_{t=0}^\infty$ is weakly increasing, we have $\min BR(\psi_-^t|x) \geq \min BR(\psi_-^{t-1}|x)$ by the supermodularity. Hence, $\min BR(\psi_-^t|x) \geq \psi_-^t(x)$.

Now let $\sigma_-^t = \psi_-^{t+T}$ for $t \geq 0$. Then $(\sigma_-^t)_{t=0}^\infty$ satisfies properties (1)–(5). In particular, along the sequential best response sequence $(\psi_-^t)_{t=0}^\infty$, at most T players change actions by period T, so that $\sigma_-^0(x) = \psi_-^T(x) = 0$ execpt for finitely many x. The construction of $(\sigma_+^t)_{t=0}^\infty$ is analogous.

For property (6), pick any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$. For each $x \in \mathcal{X}$, let $\underline{\tilde{\sigma}}^t(x) = \inf_{\tau \geq t} \tilde{\sigma}^\tau(x)$, and $\tilde{\sigma}_-(x) = \liminf_{t\to\infty} \tilde{\sigma}^t(x)$ $(= \lim_{t\to\infty} \underline{\tilde{\sigma}}^t(x))$.

Claim 1. $\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in \mathcal{X}$.

Proof. Fix any $x \in \mathcal{X}$. By the supermodularity, we have $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\underline{\tilde{\sigma}}^t|x)$ for all $t \geq 0$. Therefore, we have

$$\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) \geq \liminf_{t\to\infty} \min BR(\underline{\tilde{\sigma}}^t|x)$$

$$\geq \min BR\left(\lim_{t\to\infty} \underline{\tilde{\sigma}}^t \mid x\right) = \min BR(\tilde{\sigma}_-|x),$$

where the second inequality follows from the lower semicontinuity of $\min BR(\cdot|x)$ in the product topology on $S^{\mathcal{X}}$. ▮

Claim 2. *$\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$ for all $x \in \mathcal{X}$.*

Proof. Fix any $x \in \mathcal{X}$. By Claim 1, there exists $T_1 \geq 0$ such that $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ for all $t \geq T_1$. By (ii) and (iii) in Definition A.1, there exists $T_2 \geq T_1$ such that $\tilde{\sigma}^{T_2}(x) \geq \min BR(\tilde{\sigma}_-)$. By (ii) in Definition A.1, we also have $\tilde{\sigma}^t(x) \geq \tilde{\sigma}^{T_2}(x) \wedge \min_{T_2 \leq \tau < t} \min BR(\tilde{\sigma}^\tau|x)$ for all $t \geq T_2$. Therefore, by Claim 1 it follows that $\tilde{\sigma}^t(x) \geq \min BR(\tilde{\sigma}_-)$ for all $t \geq T_2$, and hence $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$, as desired. ▮

Claim 3. *$\sigma_-^t \leq \tilde{\sigma}_-$ for all $t \geq 0$.*

Proof. We proceed by induction. First, we want to show $\sigma_-^0 \leq \tilde{\sigma}_-$. By assumption, $\sigma_-^0 \leq \tilde{\sigma}^0$. Assume that $\sigma_-^0 \leq \tilde{\sigma}^{t-1}$, and consider any $x \in \mathcal{X}$ such that $\tilde{\sigma}^t(x) \neq \tilde{\sigma}^{t-1}(x)$. Then by the property (5) and the supermodularity, $\sigma_-^0(x) \leq \min BR(\sigma_-^0|x) \leq \min BR(\tilde{\sigma}^{t-1}|x) \leq \tilde{\sigma}^t(x)$. Therefore, we have $\sigma_-^0 \leq \tilde{\sigma}^t$ for all $t \geq 0$, and hence $\sigma_-^0 \leq \tilde{\sigma}_-$.

Next, assume that $\sigma_-^{t-1} \leq \tilde{\sigma}_-$, and let $x \in \mathcal{X}$ be such that $\sigma_-^t(x) \neq \sigma_-^{t-1}(x)$. Then by the property (3), the induction hypothesis, the supermodularity, and Claim 2, we have $\sigma_-^t(x) = \min BR(\sigma_-^{t-1}|x) \leq \min BR(\tilde{\sigma}_-|x) \leq \tilde{\sigma}_-(x)$. Thus we have $\sigma_-^t \leq \tilde{\sigma}_-$. ▮

Symmetrically, defining $\tilde{\sigma}_+(x) = \limsup_{t\to\infty} \tilde{\sigma}^t(x)$, we can show that $\tilde{\sigma}_+ \leq \sigma_+^t$ for all $t \geq 0$. For each $x \in \mathcal{X}$, since $\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$, we have $\tilde{\sigma}_-(x) = \tilde{\sigma}_+(x) = s^*$, and hence $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$.

This completes the proof of Lemma A.1. ▮

Proposition A.1. *Fix a network $(\mathcal{X}, P)$ and a supermodular game u. Then s^* is contagious by sequential best responses in $(\mathcal{X}, P)$ if and only if it is contagious by generalized best responses in $(\mathcal{X}, P)$.*

Proof. The "if" part holds by definition. To show the "only if" part, suppose that s^* is contagious by sequential best responses in $(\mathcal{X}, P)$. Let $(\sigma_-^t)_{t=0}^\infty$ and $(\sigma_+^t)_{t=0}^\infty$ be sequential best response sequences as in Lemma A.1. Let Y be a finite subset of $\mathcal{X}$ such that $\sigma_-^0(x) = 0$ and $\sigma_+^0(x) = n$ for all $x \in \mathcal{X} \setminus Y$. Then for any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\tilde{\sigma}^0(x) = s^*$ for all $x \in Y$, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$, and hence by Lemma A.1, $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ for all $x \in \mathcal{X}$. Thus s^* is contagious by generalized best responses in $(\mathcal{X}, P)$. ▮

Similarly, we can prove the equivalence between contagion by simultaneous best responses and contagion by generalized best responses. Here we assume that the set of neighbors $\Gamma(x)$ is finite for each player $x \in \mathcal{X}$, which is satisfied in all of our examples.

Proposition A.2. *Fix a network $(\mathcal{X}, P)$ such that $\Gamma(x)$ is finite for each $x \in \mathcal{X}$ and a supermodular game u. Then s^* is contagious by simultaneous best responses in $(\mathcal{X}, P)$ if and only if it is contagious by generalized best responses in $(\mathcal{X}, P)$.*

Proof. The "if" part holds by definition. The proof of the "only if" part is to mimic the proofs of Lemma A.1 and the "only if" part of Proposition A.1. Indeed, as in the proof of Lemma A.1, we take a simultaneous best response sequence $(\phi^t_-)_{t=0}^\infty$, modify it to obtain a generalized (not necessarily simultaneous) best response sequence $(\psi^t_-)_{t=0}^\infty$, and then define $(\sigma^t_-)_{t=0}^\infty$ by $\sigma^t_- = \psi^{t+T}_-$ for sufficiently large T. The only difference lies here, where it is not the case in general that "at most T players change actions by period T". Instead, we first assume without loss of generality that action 0 (as well as action n) is a Nash equilibrium of u, and resort to the finiteness of $\Gamma(x)$ to show that in each step of $(\psi^t_-)_{t=0}^T$, only finitely many players have minimum best responses other than action 0. ▮

Another definition is to only require *some* sequential best response sequence to converge.

Definition A.3. Action s^* is *weakly contagious in network* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that for any initial action configuration σ^0 such that $\sigma^0(x) = s^*$ for any $x \in Y$, there exists a sequential best response sequence (σ^t) such that $\lim_{t\to\infty} \sigma^t(x) = s^*$ for any $x \in \mathcal{X}$.

By definition, contagion implies weak contagion. The converse does not always hold. A counterexample is given by the trivial payoff function $u \equiv 0$, where all actions are weakly contagious but none of them is contagious. Nevertheless, we can show that weak contagion is equivalent to contagion for generic supermodular games.

We say that a game u is *generic* for $(\mathcal{X}, P)$ if no player has multiple best responses to any action configuration on $(\mathcal{X}, P)$. If each player has finitely many neighbors, then genericity excludes at most countably many hyperplanes in the payoff parameter space.

Proposition A.3. *Fix a network $(\mathcal{X}, P)$ and a generic supermodular game u for $(\mathcal{X}, P)$. Then s^* is weakly contagious in $(\mathcal{X}, P)$ if and only if it is contagious by generalized best responses in $(\mathcal{X}, P)$.*

Proof. Once again, the proof is almost the same as the proofs of Lemma A.1 and Proposition A.1. We only need to make the following two changes.

First, in the first paragraph of the proof of Lemma A.1, given a finite set $Y \subset \mathcal{X}$ as in Definition A.3, let $(\phi_-^t)_{t=0}^\infty$ be a sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = 0$ for all $x \in \mathcal{X} \setminus Y$, and $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in \mathcal{X}$. Here it follows from the genericity of u that we have $\phi_-^t(x) \in \{\phi_-^{t-1}(x), BR(\phi_-^{t-1}|x)\}$ for any $x \in \mathcal{X}$ and $t \geq 1$, where with an abuse of notation, $BR(\phi_-^{t-1}|x)$ denotes the unique best response.

Second, a weakly contagious action is always a Nash equilibrium, but may not be a *strict* Nash equilibrium in all games. Here again, the genericty assumption guarantees that the weakly contagious action s^* is a strict Nash equilibrium. ▮

A.2 Proof of Lemma 2

Recall

$$
\begin{aligned}
e^* &= \frac{(a-d)(d-b)}{2(c-b)}, \\
e^{**} &= \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}, \\
e^\sharp &= \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)},
\end{aligned}
$$

where $e^* \lesseqgtr e^{**} \lesseqgtr e^\sharp$ if $c - b \lesseqgtr a - c$.

Proof of Lemma 2. (1) We first note that for all $p \in (0, 1/2)$, $u(2, \pi^b) > u(0, \pi^b)$ and hence $0 \notin br(\pi^b)$.

We divide the argument into two cases: (α) $e > (a-c)/2$ and (β) $e \leq (a-c)/2$.

(α) $e > (a-c)/2$: In this case, if we let $p = 0$ (hence $\pi^a = \pi^b$), $br(\pi^a) = br(\pi^b) = \{2\}$, and thus condition (4.2) holds for some $p \in (0, 1/2)$ close to 0 due to the upper semi-continuity of br.

(β) $e \leq (a-c)/2$: In this case, for all $p \in (0, 1/2)$, $u(1, \pi^a) > u(2, \pi^a)$ and hence $2 \notin br(\pi^a)$. Therefore, $\max br(\pi^a) = 0 \Leftrightarrow u(0, \pi^a) > u(1, \pi^a)$ and $\max br(\pi^b) \leq 1 \Leftrightarrow u(1, \pi^b) > u(2, \pi^b)$, while $\min br(\pi^a) \geq 1 \Leftrightarrow u(1, \pi^a) > u(0, \pi^a)$ and $\min br(\pi^b) = 2 \Leftrightarrow u(2, \pi^b) > u(1, \pi^b)$.

Verify that $e^* \leq (a-c)/2$ with the equality holding if and only if $c = d$. Consider first the case where $e^* < (a-c)/2$ (or $c < d$). Then, since

$$
\begin{aligned}
u(0, \pi^a) - u(1, \pi^a) &= (d-b)\left\{p - \frac{(d-b)-2e}{2(d-b)}\right\}, \\
u(1, \pi^b) - u(2, \pi^b) &= (d-c)\left\{\frac{(a-c)-2e}{2(d-c)} - p\right\},
\end{aligned}
$$

it follows that condition (4.1) holds for some $p \in (0, 1/2)$ if and only if

$$
\frac{(d-b)-2e}{2(d-b)} < \frac{(a-c)-2e}{2(d-c)} \iff e < e^*,
$$

while condition (4.2) holds for some $p \in (0, 1/2)$ if and only if

$$\frac{(a-c)-2e}{2(d-c)} < \frac{(d-b)-2e}{2(d-b)} \iff e > e^*.$$

If $e^* = (a-c)/2$ (or $c = d$), then $u(0, \pi^a) > u(1, \pi^a)$ and $u(1, \pi^b) > u(2, \pi^b)$ for some $p \in (0, 1/2)$ close to 1/2 whenever $e < e^*$. (The condition $e > e^*$ never holds in the current case of $e \leq (a-c)/2$ $(= e^*)$.)

(2) We first note that $u(2, \pi^d) > u(0, \pi^d)$ and hence $0 \notin br(\pi^d)$ for all $r \in (0, 1)$. Therefore,

$$\begin{aligned} \max br(\pi^d) \leq 1 &\iff u(1, \pi^d) > u(2, \pi^d) \\ &\iff r < \frac{(a-c)-2e}{a-c}. \end{aligned} \tag{A.2}$$

For the last inequality to hold, it is necessary that $e < (a-c)/2$.

Under the condition that $e < (a-c)/2$, note that $u(1, \pi^c) > u(2, \pi^c)$ and hence $2 \notin br(\pi^c)$ for all $q \in (0, 1)$. Therefore,

$$\begin{aligned} \max br(\pi^c) = 0 &\iff u(0, \pi^c) > u(1, \pi^c) \\ &\iff q > \frac{(d-b)-2e}{d-b}. \end{aligned} \tag{A.3}$$

Finally,

$$\begin{aligned} &\max br(\pi^e) = 0 \\ &\iff u(0, \pi^e) > u(1, \pi^e) \text{ and } u(0, \pi^e) > u(2, \pi^e) \\ &\iff r > \frac{(d-b)-2e}{d-b} q \text{ and } r > \frac{(d-b)-(a-d)}{a-b} q. \end{aligned} \tag{A.4}$$

From (A.2)–(A.4), it follows that condition (4.3) holds for some $0 < r \leq q < 1$ if and only if

$$\begin{cases} \dfrac{(a-c)-2e}{a-c} > \left\{ \dfrac{(d-b)-2e}{d-b} \right\}^2 \\ \dfrac{(a-c)-2e}{a-c} > \dfrac{(d-b)-(a-d)}{a-b} \cdot \dfrac{(d-b)-2e}{d-b}, \end{cases}$$

which reduces to $e < \min\{e^\sharp, e^{**}\}$. ▮

A.3 Proof of Lemma 3

We show a stronger result, that a strict MP-maximizer is uninvadable with respect to a wider class of best response sequences. We write BR_f for the best correspondence in the local interaction game $(\mathcal{X}, P, f)$:

$$BR_f(\sigma|x) = \big\{ s \in S \;\big|\; \textstyle\sum_{y \in \Gamma(x)} P(y|x) f(s, \sigma(y))$$

$\geq \sum_{y\in\Gamma(x)} P(y|x)f(s',\sigma(y))$ for all $s' \in S\big\}$.

(Thus the best response correspondence for u as defined in (2.2) is now denoted BR_u.) Recall that $BR_f(\sigma|x) = br_f(\pi(\sigma|x))$. We consider sequences that satisfy the following property.

Definition A.4. Given a local interaction system $(\mathcal{X}, P)$ and for a payoff function $f\colon S \times S \to \mathbb{R}$, a sequence $(\sigma^t)_{t=0}^{\infty}$ satisfies property $\mathbf{B}^*$ in f if for each $t \geq 1$, there exists $x^t \in \mathcal{X}$ such that $\sigma^t(x^t) \in BR_f(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Best response sequences as defined in Definition 1 clearly satisfy this property (with $f = u$).

Let s^* be a strict MP-maximizer of u with a strict MP-function v. Recall that v is a symmetric function (i.e., $v(h,k) = v(k,h)$). The game defined by a symmetric function v is called a *potential game*, and given that $\{(s^*,s^*)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, $s^* \in S$ is called a *potential maximizer* of v. The following result is due to Morris (1999, Proposition 6.1). We provide its proof for completeness.

Lemma A.2. *Suppose that s^* is a potential maximizer of a potential game v. For any unbounded local interaction system $(\mathcal{X}, P)$ and for any sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0_P(S\setminus\{s^*\}) < \infty$ that satisfies property $\mathbf{B}^*$ in v, there exists $M < \infty$ such that $\sigma^t_P(S\setminus\{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let s^* be a potential maximizer of a potential game v. Let $\bar{\gamma} = \max_{h,k}\big(v(s^*,s^*) - v(h,k)\big) < \infty$ and $\underline{\gamma} = \min_{(h,k)\neq(s^*,s^*)}\big(v(s^*,s^*) - v(h,k)\big) > 0$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any sequence such that $\sigma^0_P(S\setminus\{s^*\}) < \infty$, and assume that it satisfies property $\mathbf{B}^*$ in v. Let $(x^t)_{t=1}^{\infty}$ be such that $\sigma^t(x^t) \in BR_v(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Let

$$V(t) = \frac{1}{2}\sum_{(x,y)\in\mathcal{X}\times\mathcal{X}} P(x,y)\big(v(\sigma^t(x),\sigma^t(y)) - v(s^*,s^*)\big).$$

Note that

$$-\bar{\gamma}\sigma^t_P(S\setminus\{s^*\}) \leq V(t) \leq -\underline{\gamma}\sigma^t_P(S\setminus\{s^*\}).$$

Since $\sigma^0_P(S\setminus\{s^*\}) < \infty$, we have $V(0) > -\infty$. Also we have

$$\begin{aligned} &V(t) - V(t-1)\\ &\qquad= \sum_{y\in\Gamma(x^t)} P(x^t,y)\big(v(\sigma^t(x^t),\sigma^{t-1}(y)) - v(\sigma^{t-1}(x^t),\sigma^{t-1}(y))\big) \geq 0 \end{aligned}$$

by property $\mathbf{B}^*$. It follows from the induction on t that V is nondecreasing, so that $V(t) \geq V(0)$ for all t.

Then we have $\sigma^t_P(S\setminus\{s^*\}) \leq -V(t)/\underline{\gamma} \leq -V(0)/\underline{\gamma}$ for all t. ∎

Lemma 3 is a direct corollary of the following.

Lemma A.3. *Suppose that s^* is a strict MP-maximizer of u with a strict MP-function v. If u or v is supermodular, then for any unbounded local interaction system $(\mathcal{X}, P)$ and for any sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma_P^0(S \setminus \{s^*\}) < \infty$ that satisfies property $\boldsymbol{B}^*$ in u, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let $s^* = \underline{s}, \overline{s}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any sequence such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$, and assume that it satisfies property $\mathbf{B}^*$ in u. Let $(x^t)_{t=1}^{\infty}$ be such that $\sigma^t(x^t) \in BR_u(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Now let $(\hat{\sigma}^t)_{t=0}^{\infty}$ be defined by $\hat{\sigma}^0 = \sigma^0$ and for $t \geq 1$,

$$\hat{\sigma}^t(x) = \begin{cases} \max BR_v(\hat{\sigma}^{t-1}|x^t) & \text{if } x = x^t, \\ \hat{\sigma}^{t-1}(x) & \text{otherwise.} \end{cases}$$

Then, $(\hat{\sigma}^t)_{t=0}^{\infty}$ satisfies $\mathbf{B}^*$ in v. Therefore, by Lemma A.2, there exists M such that $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show that if u or v is supermodular, then

$$\sigma^t(x) \leq \hat{\sigma}^t(x) \text{ for all } x \in \mathcal{X}. \qquad (\star_t)$$

for all $t \geq 0$. Then, $\sigma_P^t(S \setminus \{\underline{s}\}) \leq \hat{\sigma}_P^t(S \setminus \{\underline{s}\})$ for all t, and since $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t, it follows that $\sigma_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show by induction that $(\star_t)$ holds for all $t \geq 0$. First, $(\star_0)$ trivially holds by the definition of $\hat{\sigma}^0$. Then, assume $(\star_{t-1})$. It implies that for all $x \in \mathcal{X}$, $\pi(\sigma^{t-1}|x) \precsim \pi(\hat{\sigma}^{t-1}|x)$. By construction, $\sigma^t(x) = \hat{\sigma}^t(x)$ for all $x \neq x^t$. For $x = x^t$, if u is supermodular, then

$$\begin{aligned} \sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_u(\pi(\hat{\sigma}^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t), \end{aligned}$$

where the second inequality follows from the supermodularity of u, and the third inequality follows from the MP condition (4.5). If v is supermodular, then

$$\begin{aligned} \sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t), \end{aligned}$$

where the second inequality follows from the MP condition (4.5), and the third inequality follows from the supermodularity of v. Therefore, in each case, $(\star_t)$ holds. ∎

A.4 Proof of Lemma 4

For $f\colon S \times S \to \mathbb{R}$ and $h \in S$, let

$$\Pi_h(f) = \{\pi \in \Delta(S) \mid h \in br_f(\pi)\}.$$

Note that 0 is a strict MP-maximizer of u with MP-function v if and only if $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_2(u) \subset \Pi_2(v) \text{ and } \Pi_1(u) \subset \Pi_1(v) \cup \Pi_2(v),$$

while 2 is a strict MP-maximizer of u with MP-function v if and only if $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_0(u) \subset \Pi_0(v) \text{ and } \Pi_1(u) \subset \Pi_0(v) \cup \Pi_1(v).$$

Recall

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)},$$

and denote

$$e^\flat = \frac{(a-d)(d-b)}{a-b}.$$

Verify that $e^\flat \lesseqgtr e^* \lesseqgtr e^{**}$ if $c-b \lesseqgtr a-c$.

Lemma 4 is proved by Lemmas A.4–A.7 which follow. Lemma A.4 considers the case in which $c = d$ and $e < e^*$ $(= (a-c)/2)$; Lemma A.5 considers the cases of $e < e^*$ and $e^* < e \le e^\flat$ under the assumption that $c \neq d$; and Lemmas A.6 and A.7 cover the cases of $\max\{e^{**}, e^\flat\} < e \le (a-c)/2$ and $e > (a-c)/2$, respectively; see Figure A.1.

Lemma A.4. *Suppose that $c = d$. If $e < e^*$ $(= (a-c)/2 = (a-d)/2)$, then 0 is a strict MP-maximizer.*

Proof. Observe first that, since $e < e^* = (a-c)/2 = (a-d)/2 < (d-b)/2$,

$$u(0,k) - u(1,k) < u(1,k) - u(2,k) \tag{A.5}$$

for all $k = 0, 1, 2$. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{pmatrix} e & 0 & -\lambda e-(d-b)+e \\ 0 & -e & -\lambda e \\ -\lambda e-(d-b)+e & -\lambda e & 0 \end{pmatrix} \end{array}, \tag{A.6}$$

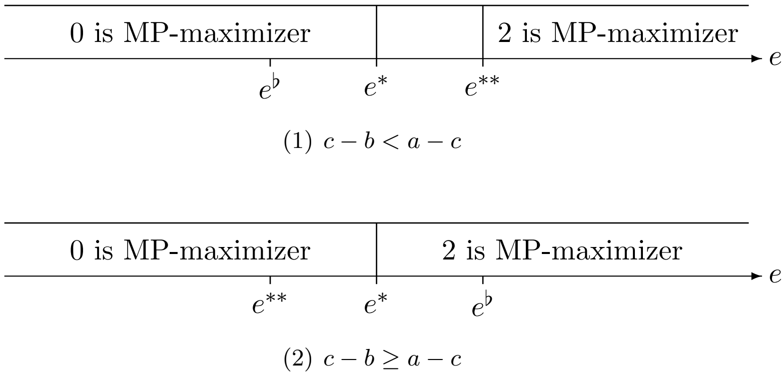


Figure A.1: MP-maximizer

where

$$\lambda = \frac{(d-b)-e}{(a-d)-2e} > 0.$$

This function is maximized at $(0,0)$. Verify that

$$v(0,k)-v(1,k) = u(0,k)-u(1,k) \tag{A.7}$$
$$v(1,k)-v(2,k) \leq \lambda\big(u(1,k)-u(2,k)\big) \tag{A.8}$$

for all $k=0,1,2$. Then, we have $\Pi_1(u)\subset\Pi_1(v)\cup\Pi_2(v)$ by (A.7), and $\Pi_2(u)\subset\Pi_2(v)$ by (A.5), (A.7), and (A.8). ▮

Lemma A.5. *Suppose that $c\neq d$.* (i) *If $e<e^*$, then 0 is a strict MP-maximizer.* (ii) *If $e^*<e\leq e^\flat$, then 2 is a strict MP-maximizer.*

Proof. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 2\lambda e & \lambda e & \lambda e-(a-c)+e \\ \lambda e & 0 & -(a-d)+e \\ \lambda e-(a-c)+e & -(a-d)+e & -(a-d)+2e \end{pmatrix} \end{array}, \tag{A.9}$$

where

$$\lambda = \frac{d-c}{d-b} > 0.$$

We show that this function v works as a strict MP-function if $e\leq\max\{e^*,e^\flat\}$ and $e\neq e^*$.

We first have the following.

Claim 1. (i) $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e<e^*$.* (ii) $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e>e^*$.*

Verify that

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.10}$$
$$v(1,k) - v(2,k) = u(1,k) - u(2,k) \tag{A.11}$$

for all $k = 0, 1, 2$. These immediately imply the following.

Claim 2. $\Pi_1(u) = \Pi_1(v)$.

For $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$, we have

$$u(0,\pi) - u(1,\pi) = (d-b)\left(\frac{e}{d-b} - \pi_2\right), \tag{A.12}$$

$$u(1,\pi) - u(2,\pi) = (d-c)\left\{\pi_0 - \frac{(a-b)e - (a-d)(d-b)}{(d-b)(d-c)}\right\} + (a-d)\left(\frac{e}{d-b} - \pi_2\right), \tag{A.13}$$

and

$$v(2,\pi) - v(0,\pi) = \big(u(2,\pi) - u(0,\pi)\big) + (c-b)\left(\frac{e}{d-b} - \pi_2\right). \tag{A.14}$$

These imply the following.

Claim 3. $\Pi_2(u) \subset \Pi_2(v)$.

Proof. Assume that $\pi = (\pi_0, \pi_1, \pi_2) \in \Pi_2(u)$ ($\Leftrightarrow u(2,\pi) \geq u(0,\pi)$ and $u(2,\pi) \geq u(1,\pi)$). First, by (A.11), $u(2,\pi) \geq u(1,\pi)$ implies $v(2,\pi) \geq v(1,\pi)$. Second, if $\pi_2 \geq e/(d-b)$, then by (A.10) and (A.12), we have $v(1,\pi) \geq v(0,\pi)$ and therefore $v(2,\pi) \geq v(0,\pi)$, while if $\pi_2 < e/(d-b)$, then by (A.14), $u(2,\pi) \geq u(0,\pi)$ implies $v(2,\pi) > v(0,\pi)$. We thus have $\pi \in \Pi_2(v)$. ▮

Claim 4. *If $e \leq e^\flat$, then $br_u = br_v$.*

Proof. Suppose that $e \leq e^\flat$. In light of Claim 2, we want to show that $\Pi_0(u) = \Pi_0(v)$ and $\Pi_2(u) = \Pi_2(v)$.

Note in (A.13) that $e \leq e^\flat$ implies $\{(a-b)e - (a-d)(d-b)\}/\{(d-b)(d-c)\} \leq 0$. By (A.12) and (A.13), we therefore have $u(0,\pi) \geq u(1,\pi) \Rightarrow u(1,\pi) \geq u(2,\pi)$ and $u(2,\pi) \geq u(1,\pi) \Rightarrow u(1,\pi) \geq u(0,\pi)$. By (A.10) and (A.11), it thus follows that $\pi \in \Pi_0(u) \Leftrightarrow u(0,\pi) \geq u(1,\pi) \Leftrightarrow v(0,\pi) \geq v(1,\pi) \Leftrightarrow \pi \in \Pi_0(v)$ and $\pi \in \Pi_2(u) \Leftrightarrow u(2,\pi) \geq u(1,\pi) \Leftrightarrow v(2,\pi) \geq v(1,\pi) \Leftrightarrow \pi \in \Pi_2(v)$. ▮

We now complete the proof of Lemma A.5. (i) If $e < e^*$, Claims 1, 2, and 3 imply that 0 is a strict MP-maximizer. (ii) If $e^* < e \leq e^\flat$, Claims 1 and 4 imply that 2 is a strict MP-maximizer. ▮

Lemma A.6. *If* $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$*, then* 2 *is a strict MP-maximizer.*

Proof. Suppose that $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$. Let v be defined by

$$\begin{array}{c} \\ 0 \\ \\ 1 \\ \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \left(\begin{array}{ccc} 0 & -\lambda e & \begin{array}{l} -\lambda e \\ \quad -\{(a-c)-e\} \end{array} \\ -\lambda e & -2\lambda e & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-e\} \end{array} \\ \begin{array}{l} -\lambda e \\ \quad -\{(a-c)-e\} \end{array} & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-e\} \end{array} & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-2e\} \end{array} \end{array}\right) \end{array}, \tag{A.15}$$

where

$$\lambda = \frac{(a-c)(d-b)-(a-b)e}{(a-b)\{(d-b)-e\}} > 0$$

($\lambda > 0$ follows from $e \leq (a-c)/2$). We show that this function v works as a strict MP-function.

First, the function (A.15) is maximized at $(2,2)$ (by $e > e^{**}$). Second, one can verify, for all $k = 0, 1, 2$,

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.16}$$

$$v(0,k) - v(2,k) \geq \frac{a-c}{a-b}\big(u(0,k) - u(2,k)\big) \tag{A.17}$$

(by $e \leq (a-c)/2$), and

$$v(1,k) - v(2,k) \geq u(1,k) - u(2,k) \tag{A.18}$$

(since $\lambda < (d-c)/(d-b)$ by $e > e^\flat$). Therefore, $\pi \in \Pi_0(u) \Rightarrow \pi \in \Pi_0(v)$ by (A.16)–(A.17) and $\pi \in \Pi_1(u) \Rightarrow \pi \in \Pi_0(v) \cup \Pi_1(v)$ by (A.18). ▮

Lemma A.7. *If* $e > (a-c)/2$*, then* 2 *is a strict MP-maximizer.*

Proof. Action 2 is strictly p-dominant with

$$p = \max\left\{\frac{a-c-e}{a-c}, \frac{a-c}{(a-c)+(d-b)}\right\},$$

i.e., $\{2\} = br_u(\pi)$ for any $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$ such that $\pi_2 > p$ (Morris et al. (1995), Kajii and Morris (1997)). If $e > (a-c)/2$, we have $p < 1/2$. Therefore, the function

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & -p \\ 0 & 0 & -p \\ -p & -p & 1-2p \end{array}\right) \end{array} \tag{A.19}$$

is a strict MP-function for 2 (see Morris and Ui (2005) or Oyama et al. (2008, Lemma 4.1)). ▮

A.5 Proof of Theorem 2

Since the network used in the proof of Lemma 1(1) is linear, combined with Lemma 2(1) it follows that if $e < e^*$ (resp. $e > e^*$), then action 0 (resp. 2) is contagious in linear networks. Also, by Proposition 2(i), if $e < e^*$, then 0 is uninvadable, hence uninvadable in linear networks. Thus, we only need to show that 2 is uninvadable in linear networks if $e > e^*$.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.2). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\min br(\tilde{\pi}^a) \geq 1$ and $\min br(\tilde{\pi}^b) = 2$, where

$$\tilde{\pi}^a = \left(\tfrac{1}{2} + \varepsilon, p, \tfrac{1}{2} - p - \varepsilon\right), \quad \tilde{\pi}^b = \left(\tfrac{1}{2} - p + \varepsilon, p, \tfrac{1}{2} - \varepsilon\right).$$

Fix any linear network $(\mathbb{Z}, P)$. Since $P(0|0) = 0$ and $P(y|0) = P(-y|0)$ for all $y > 0$, we have $\sum_{y=1}^{\infty} P(y|0) = 1/2$. Let n_1 be the smallest integer such that $\sum_{y=1}^{n_1} P(y|0) \geq p$, and n_2 be a sufficiently large integer such that $\sum_{y>n_2} P(y|0) \leq \varepsilon$.

Consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(\{0,1\}) < \infty$. Let K be the set of all $k \in \mathbb{Z}$ such that $\sigma^0(x) = 2$ if $|x - k| \leq n_1 + n_2$. Then K is co-finite (i.e., $\mathbb{Z} \setminus K$ is finite), and so is $L = \bigcup_{k \in K} \{x \in \mathbb{Z} \mid |x - k| \leq n_2\}$. (Otherwise, $\sigma^0(x) \neq 2$ for infinitely many x, which contradicts the finiteness of $\sigma_P^0(\{0,1\})$.)

For each $k \in K$, we want to show that

$$\begin{aligned} &\sigma^t(x) = 2 \text{ if } |x - k| \leq n_2, \\ &\sigma^t(x) \geq 1 \text{ if } n_2 + 1 \leq |x - k| \leq n_1 + n_2 \end{aligned}$$

for all $t \geq 0$. First, this holds for $t = 0$ by construction. Next, assume that it holds for $t - 1$. Then,

- for players x such that $|x - k| \leq n_2$,

  $$\pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} - \varepsilon,$$

  $$\pi(\sigma^{t-1}|x)(1) + \pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_1} P(y|0) + \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} + p - \varepsilon,$$

  which implies that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^b$ and hence $\sigma^t(x) = 2$;

- for players x such that $n_2 + 1 \leq |x - k| \leq n_1 + n_2$,

  $$\pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_2} P(y|0) - \sum_{y=1}^{n_1 - 1} P(y|0) > \frac{1}{2} - p - \varepsilon,$$

$$\pi(\sigma^{t-1}|x)(1) + \pi(\sigma^{t-1}|x)(2) \geq \sum_{y=1}^{n_1} P(y|0) \geq \frac{1}{2} - \varepsilon,$$

which implies that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^a$ and hence $\sigma^t(x) \geq 1$.

Therefore, $\{x \in \mathbb{Z} \mid \sigma^t(x) = 2\} \supset L$, and hence $\sigma_P^t(\{0,1\})$ is bounded from above. ▌

A.6 Proof of Theorem 4

We fix the dimension m. A sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on the m-dimensional lattice $\mathbb{Z}^m$ is *well-behaved* if the following conditions are satisfied.

- For each n, P_n is invariant up to translation, i.e., $P_n(x,y) = P_n(x+z, y+z)$ for $x, y, z \in \mathbb{Z}^m$.
- There exist a pair of nonnegative integrable functions $f, g \colon \mathbb{R}^m \to \mathbb{R}_+$ such that for almost every $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$,

  $$n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0) \to f(\nu)$$

  as $n \to \infty$ (pointwise convergence), and

  $$n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0) \leq g(\nu)$$

  for every n.[16]
- f has connected support.

For example, the sequence of n-max distance interactions is well-behaved since $n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0)$ converges to 2^{-m} times the indicator function of $\{\nu \in \mathbb{R}^m \mid \max_i \nu_i \leq 1\}$.

The next result characterizes contagious and uninvadable actions in any well-behaved sequence of multidimensional lattice networks. Theorem 4 follows as an immediate corollary.

Theorem A.1. *Let u be the bilingual game given by* (3.1)*. Fix the dimension m and a well-behaved sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on $\mathbb{Z}^m$.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, 0 is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, 2 is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.*

[16]For $\eta \in \mathbb{R}$, $[\eta]$ denotes the largest integer that does not exceed η.

Proof. We will show (i) only. The proof for (ii) is analogous.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.1). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\max br(\hat{\pi}^a) = 0$ and $\max br(\hat{\pi}^b) \leq 1$, where

$$\hat{\pi}^a = \left(\tfrac{1}{2} - \varepsilon, p, \tfrac{1}{2} - p + \varepsilon\right), \quad \hat{\pi}^b = \left(\tfrac{1}{2} - p - \varepsilon, p, \tfrac{1}{2} + \varepsilon\right).$$

Let $f(\nu)$ be the pointwise limit of $n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0)$ as $n \to \infty$. Since P_n is symmetric and translation invariant, f is symmetric, i.e., $f(\nu) = f(-\nu)$ for almost all ν. We also have $\int_{\mathbb{R}^m} f(\nu) d\nu = 1$.

By the symmetry of f and the connectedness of the support of f, for each $\lambda \in \mathbb{R}^m$ whose Euclidean norm $\|\lambda\|$ is 1, there exists a unique $\delta(\lambda) > 0$ such that

$$\int_{0 \leq \lambda \cdot x \leq \delta(\lambda)} f(x) dx = p.$$

Note that $\delta(\lambda)$ is continuous in λ.

For each positive real number r, let $B_r = \{\nu \in \mathbb{R}^m \mid \|\nu\| \leq r\}$ and $C_r = \{\nu \in \mathbb{R}^m \mid r < \|\nu\| \leq r + \delta(\nu/\|\nu\|)\}$. Note that for large r and any boundary point ν of B_r, the boundary of B_r near ν is approximately "flat" and orthogonal to ν. By the continuity of $\delta(\cdot)$, the same is true for the boundary of C_r. Thus, there exists r_1 such that for any $r \geq r_1$,

$$\int_{B_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \geq \frac{1}{2} + p - \frac{\varepsilon}{3}$$

if $\nu \in B_r$, and

$$\int_{B_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - p - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}$$

if $\nu \in C_r$.

For each integer k, let $\hat{B}_k = \{x \in \mathbb{Z}^m \mid \|x\| \leq k\}$ and $\hat{C}_{k,n} = \{x \in \mathbb{Z}^m \mid k < \|x\| \leq k + n\delta(x/\|x\|)\}$. Since $(P_n)_{n=0}^{\infty}$ is well-behaved, one can apply the dominated convergence theorem to show that there exists n_1 such that for any $n \geq n_1$,

$$\left| \sum_{y \in \hat{B}_k} P_n(y - x|0) - \int_{B_{k/n}} f(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3},$$

$$\left| \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y - x|0) - \int_{B_{k/n} \cup C_{k/n}} f(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3}$$

for any $x \in \mathbb{Z}^m$ and k. Therefore, there exists $n_2 \geq n_1$ such that for any $n \geq n_2$ and any $k \geq r_1 n$,

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} + p - \varepsilon$$

for any $x \in \hat{B}_{k+1}$, and

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - p - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} - \varepsilon$$

for any $x \in \hat{C}_{k+1,n}$.

Now let $n \geq n_2$. We show that 0 is contagious in $(\mathbb{Z}^m, P_n)$. The proof is similar to that of Lemma 1(1). Pick an integer $K \geq r_1 n$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in \hat{B}_K \cup \hat{C}_{K,n}$. Then one can show by induction on k that for any $k \geq K$, there exists T_k such that for any $T \geq T_k$, we have $\sigma^t(x) = 0$ for all $x \in \hat{B}_k$ and $\sigma^0(x) \leq 1$ for all $x \in \hat{C}_{k,n}$.

This argument also shows that 0 is uninvadable in $(\mathbb{Z}^m, P_n)$ because for any initial configuration that satisfies $\sigma^0_{P_n}(\{1,2\}) < \infty$, there exists a translation Y of $\hat{B}_K \cup \hat{C}_{K,n}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. ▮

A.7 Proof of Theorem 5

We prove Theorem 5 for general supermodular games with action set $S = \{0, \ldots, n\}$.

Let φ be a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\hat{\mathcal{X}}, \hat{P})$ with a finite set E of exceptional nodes. Fix a supermodular game u, and assume that s^* is contagious in $(\mathcal{X}, P)$ (and hence a strict Nash equilibrium). We show that s^* is contagious in $(\hat{\mathcal{X}}, \hat{P})$.

Let $F \supset E$ be a sufficiently large finite subset of $\mathcal{X}$ such that for any $\hat{x} \in \varphi(E)$, the unique best response for player $\hat{x}$ is s^* if all players in $\varphi(F)$ play action s^*. (Since s^* is a strict Nash equilibrium, we can find such a finite set even if some player $\hat{x} \in \varphi(E)$ has infinitely many neighbors.)

Let $(\sigma^t_-)_{t=0}^{\infty}$ and $(\sigma^t_+)_{t=0}^{\infty}$ be sequential best response sequence in $(\mathcal{X}, P)$ that satisfy properties (1)–(5) in Lemma A.1. Pick a $T \geq 0$ such that $\sigma^T_-(x) = \sigma^T_+(x) = s^*$ for all $x \in F$, and let $Y = \{x \in \mathcal{X} \mid \sigma^T_-(x) \neq 0 \text{ or } \sigma^T_+(x) \neq n\}$. Note that Y is finite.

Define action configurations $\hat{\sigma}_-$ and $\hat{\sigma}_+$ in $(\hat{\mathcal{X}}, \hat{P})$ by

$$\hat{\sigma}_-(\hat{x}) = \max_{x \in \varphi^{-1}(\hat{x})} \sigma^T_-(x) \text{ and } \hat{\sigma}_+(\hat{x}) = \min_{x \in \varphi^{-1}(\hat{x})} \sigma^T_+(x)$$

for all $\hat{x} \in \hat{\mathcal{X}}$. Note that $\hat{\sigma}_-(\hat{x}) = \hat{\sigma}_+(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$, and $\hat{\sigma}_-(\hat{x}) = 0$ and $\hat{\sigma}_+(\hat{x}) = n$ for all $\hat{x} \in \hat{\mathcal{X}} \setminus \varphi(Y)$. Denote by $\widehat{BR}$ the set of best responses defined in $(\hat{\mathcal{X}}, \hat{P})$.

Claim 1. *$\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$ and $\hat{\sigma}_+(\hat{x}) \leq \max \widehat{BR}(\hat{\sigma}_+|\hat{x})$ for all $\hat{x} \in \hat{\mathcal{X}}$.*

Proof. We only show the first inequality; the proof of the second is anologous. For any $\hat{x} \in \varphi(E)$, since $\hat{\sigma}_-(\hat{y}) = s^*$ for all $\hat{y} \in \varphi(F)$, we have $BR(\hat{\sigma}_-|\hat{x}) = \{s^*\}$ by the construction of F. Consider next any $\hat{x} \in \mathcal{X} \setminus \varphi(E)$. Write $\bar{\sigma}_-^T$ for the action configuration in $(\mathcal{X}, P)$ given by $\bar{\sigma}_-^T(y) = \hat{\sigma}_-(\hat{y})$ if $y \in \varphi^{-1}(\hat{y})$, and let $\bar{x} \in \arg\max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x)$. Then we have $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) = \min BR(\bar{\sigma}_-^T|\bar{x}) \geq BR(\sigma_-^T|\bar{x}) \geq \sigma_-^T(\bar{x}) = \hat{\sigma}_-(\hat{x})$, where the first equality follows from the weight preserving property of φ, the first inequality from the supermodularity, and the second inequality from property (5) in Lemma A.1. ▮

Let $\hat{Y} = \varphi(Y)$, which is finite. Pick any sequential best response sequence $(\hat{\sigma}^t)$ in $(\hat{\mathcal{X}}, \hat{P})$ such that $\hat{\sigma}^0(\hat{x}) = s^*$ for all $\hat{x} \in \hat{Y}$. We want to show that $\lim_{t\to\infty} \hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \hat{\mathcal{X}}$.

Claim 2. *$\hat{\sigma}_- \leq \hat{\sigma}^t \leq \hat{\sigma}_+$ for all $t \geq 0$.*

Proof. We only show the first inequality; the proof of the second is anologous. First we have $\hat{\sigma}^0 \geq \hat{\sigma}_-$ by construction. Next assume $\hat{\sigma}^{t-1} \geq \hat{\sigma}_-$. If $\hat{\sigma}^t(\hat{x}) \neq \hat{\sigma}^{t-1}(\hat{x})$, then we have $\hat{\sigma}^t(\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}^{t-1}|\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$, where the first inequality follows from the supermodularity and the second from Claim 1. ▮

Claim 2 implies in particular that $\hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$ and all $t \geq 0$.

Given the sequence $(\hat{\sigma}^t)_{t=0}^\infty$ in $(\hat{\mathcal{X}}, \hat{P})$, let $(\tilde{\sigma}^t)_{t=0}^\infty$ be the corresponding sequence in $(\mathcal{X}, P)$ defined by

$$\tilde{\sigma}^t(x) = \hat{\sigma}^t(\varphi(x))$$

for all $x \in \mathcal{X}$ and $t \geq 0$. First, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$ since by Claim 2, $\sigma_-^0(x) \leq \sigma_-^T(x) \leq \hat{\sigma}_-(\varphi(x)) \leq \hat{\sigma}^0(\varphi(x)) \leq \hat{\sigma}_+(\varphi(x)) \leq \sigma_+^T(x) \leq \sigma_+^0(x)$ for all $x \in \mathcal{X}$. Second, $(\tilde{\sigma}^t)_{t=0}^\infty$ is a generalized best response sequence in $(\mathcal{X}, P)$ as defined in Definition A.1. (Notice that players in $\varphi^{-1}(\hat{x})$ change actions simultaneously.) Indeed, for $x \in \mathcal{X} \setminus E$, we have $BR(\tilde{\sigma}^t|x) = \widehat{BR}(\hat{\sigma}^{t-1}|\varphi(x))$ for all $t \geq 0$ by the weight preserving property of φ, while for $x \in E$, by construction we have $\tilde{\sigma}^t(x) = \sigma_-^{t+T}(x) = \sigma_+^{t+T}(x) = s^*$ and $BR(\sigma_-^{t+T}|x) = BR(\tilde{\sigma}^t|x) = BR(\sigma_+^{t+T}|x) = \{s^*\}$ for all $t \geq 0$. Thus, by Lemma A.1(6), $(\sigma^t(x))_{t=0}^\infty$ converges to s^* for all $x \in \mathcal{X}$, and hence $(\hat{\sigma}^t(\hat{x}))_{t=0}^\infty$ also converges to s^* for all $\hat{x} \in \hat{\mathcal{X}}$. ▮

A.8 The Case Where Pareto-Dominance and Risk-Dominance Coincide

For completeness, we report the contagion and uninvadability result also for the case where action 0 is both Pareto-dominant and pairwise risk-dominant.

The game u,

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{A.20a}$$

now satisfies

$$c \leq d < a, \quad d-b < a-c, \quad e > 0. \tag{A.20b}$$

Theorem A.2. *Let u be the bilingual game given by* (A.20). 0 *is always contagious and uninvadable.*

Proof. In light of Lemma 1(1-i) and Lemma 3, it suffices to show that condition (4.1) holds for some p and that 0 is a strict MP-maximizer. If $e \leq (d-b)/2$, we have $(c-b)e < (a-d)(d-b)/2$. Therefore, these follow from the argument in case (α) in the proof of Lemma 2(1) and Claims 1–3 in the proof of Lemma A.5. If $e > (d-b)/2$, they follow from the symmetric arguments for 0 in place of 2 as in case (β) in the proof of Lemma 2(1) and Lemma A.7. ▌

The contagion part of this theorem has been shown by Goyal and Janssen (1997, Theorem 3) in their circular network setting with a continuum of players.

Immorlica et al. (2007) consider the current case with a payoff parameter restriction $a = 1-q$, $b = c = 0$, and $d = q$, so the game is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1-q & 1-q & 0 \\ 1-q-e & 1-q-e & q-e \\ 0 & q & q \end{pmatrix} \end{array}, \qquad 0 < q < \frac{1}{2}. \tag{A.21}$$

(Note that by reversing the order of the actions, we know from Theorem A.2 that action 2 is uninvadable if $q > 1/2$.) They focus on the class $\mathcal{G}_\Delta$ of Δ-regular networks; for a natural number Δ, a Δ-regular network is a constant-weight local interaction system where each player has Δ neighbors. They consider the "epidemic region" $\Omega(G) \subset (0,1/2) \times \mathbb{R}_{++}$, the set of points (q,e) for which action 0 spreads contagiously in a network G, and show that for any Δ, there exists a point $(q,e) \notin \Omega_\Delta = \bigcup_{G \in \mathcal{G}_\Delta} \Omega(G)$, and in particular, Ω_Δ is not convex. On the other hand, since the linear network constructed in Lemma 1(1-i) (with a choice of a rational number p) can be replicated by a Δ-regular network, our Theorem A.2 implies that $\Omega^* = \bigcup_\Delta \Omega_\Delta$ covers the whole space $(0,1/2) \times \mathbb{R}_{++}$ and is convex.

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