Optimal Shirking in Teams*

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November 18, 2009

Abstract

We study a model of repeated team production with symmetric and patient agents, where the outcome is noisy and binary. A natural candidate for the second-best equilibrium is the one with a full cooperation stage; all agents are induced to work at least some histories. In contrast, we show that the second-best equilibrium sometimes has the form where at any history at least one agent is allowed to shirk. The source of this optimality of shirking is the efficiency loss from providing incentives to work to all agents simultaneously, which is avoided by equilibria where incentives are provided by transfer of the privilege to shirk among agents. The result is robust to introduction of sabotage.

JEL Classification Numbers: C72, C73, L23, P13.

* *Very Preliminary.*

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1 Introduction

The team production with several agents is subject to the free-rider problem, but the problem may be mitigated to the extent that the team forms a long-run relationship. This type of model of repeated team production has been much studied in the literature, which has shown that the efficiency of the team production depends on the richness of information about the agents' effort levels. Two important papers prove polar results. One is the *uniform inefficiency result* due to Radner, Myerson and Maskin (1986), stating that even very patient agents cannot avoid efficiency loss if the signal space is small in comparison with the effort space. The other is the *folk theorem* by Fudenberg, Levine and Maskin (1994), which shows that if the signal space is richer, then in general the efficiency loss goes to zero as the agents' discounting vanishes.

The present paper adds to the literature on uniform inefficiency results, by studying a version of the model by Radner, Myerson and Maskin (1986). Every period, n agents decide whether to work for the team or shirk, and their decisions affect a stochastic outcome of the production of that period, which is either good or bad. The agents are symmetric; their costs of efforts are the same, they evaluate good and bad outcomes in the same way, and the probability of each outcome depends only on the number of agents who work. The assumption of binary outcomes lets our model belong to the uniform inefficiency literature. Our objective is to characterize the most efficient equilibrium when the agents are sufficiently patient.[1]

Given the symmetry, one natural candidate for the second-best equilibrium is the one where all agents work (with a nonzero probability) in the initial period. The grim-trigger strategies are an example of such *strategies with initial full cooperation.* However, we show that it is often the case that equilibria by such strategies are not second-best. We present conditions under which the most efficient equilibrium is such that at any history there exists an agent who is not prescribed to work at all at that history. That is, in order to achieve the second-best, the team must give up to provide incentives to all members at any moment. Roughly speaking, the conditions for this optimality of shirking hold when the probability of good outcome is small even when all agents work.

A key insight to our result is that the signal space is binary, and therefore one cannot statistically determine the identity of a deviator from a fully cooperative action profile. A bad signal indicates a deviation, but one does not know who is the deviator. Hence all agents must be punished upon a bad signal, which causes an inevitable efficiency loss in equilibrium. Moreover, the formula by Abreu, Milgrom and Pearce (1991) gives the size of efficiency loss explicitly. If the efficiency loss is greater than the loss associated with presence of slackers every period, it is possible that the equilibria with slackers are optimal. One factor which affects the size of efficiency loss is the likelihood ratio of a bad outcome between profiles with n and $n-1$ cooperators. If the success is very unlikely, this likelihood ratio is close to 1. Hence the detectability of deviation is weak, and the efficiency loss is big. In this case, the optimality of shirking is likely to hold.

[1] Relatedly, Kobayashi, Ohta and Sekiguchi (2008) study a two-agent version of the model for any level of patience, under an additional assumption that the partners can commit to a sharing rule at the beginning of their repeated interaction.

Our methodology is to consider a particular class of strategies where at no history all agents work with a nonzero probability, and examine whether equilibria by such strategies exist. We call equilibria by those strategies *turnover equilibria with k shirkers*, where $k \in \{1, 2, \cdots, n-1\}$, whose idea is due to Rob and Sekiguchi (2006). A *turnover strategy profile with k shirkers* is an automaton strategy profile with the following property. Its state variable is an element of the sets of agents with k members. If a particular set of agents with k members is the state variable of a period, then the members in that set are prescribed to shirk in that period. The others are prescribed to work. The state variable in the next period (another set of agents with k members) depends on the outcome of this period. If the outcome is bad, the state variable remains the same. If it is good, then a *turnover* occurs according to the following rule. If $k \leq n-k$, then a set of agents with k members which is disjoint with the current set is chosen with equal probability, and it is the state variable in the next period. If $k \geq n-k$, then a set of agents with k members which includes the complement of the current set is chosen with equal probability, and it is the state variable in the next period.[2] Namely, the turnover occurs so that the maximal possible number of agents working now (which is $\max\{k, n-k\}$) is (randomly) selected and they are given the privilege to shirk in the next period.

The turnover strategy profiles with some shirkers are clearly asymmetric, but symmetry of its rules on turnover makes it tractable to check its equilibrium conditions. If a turnover strategy profile with k shirkers is an equilibrium, its (average) payoff sum is the stage-payoff sum of a profile with $n-k$ workers. If the sum is greater than the bound on equilibrium payoff sums by strategies with initial full cooperation, then those equilibria are shown not to be second-best. In this way, the optimality of shirking obtains.

This optimality result leaves possibilities that (i) there may be better equilibria than the turnover equilibrium, and (ii) the same payoff sum may be sustained by some other equilibrium, even when the equilibrium condition for the turnover equilibrium is not satisfied. However, we can show a much stronger result if there exists a turnover equilibrium with 1 shirker and it is more efficient than any other equilibrium with initial full cooperation. In this case, we show that (i) there is no other equilibrium with a greater payoff sum than the turnover equilibrium with 1 shirker, and (ii) if there exists an equilibrium with a payoff sum equal to the stage-payoff sum of a profile with $n-1$ workers, the turnover strategy profile with 1 shirker is also an equilibrium. Hence in this case, restriction to turnover strategies loses no generality.

We also present one extension of our model where the agents have an option to sabotage production. Namely, we assume that each agents has a costly option which simply reduces the probability of success. This option is potentially important because given a turnover strategy profile a shirker in a current period may benefit from failure in the current period, because of an increased probability that his privilege is preserved. However, we show that this option never affects the equilibrium condition for the turnover profiles. Consequently, the same optimality of shirking is valid in models with sabotage.

[2] The two rules are equivalent if $k = n-k$, in which case the complement of the current set is the state variable in the next period with probability 1.

The rest of this paper is organized as follows. In Section 2, we introduce the model. In Section 3, we first review Radner, Myerson and Maskin's (1986) uniform inefficiency result and Abreu, Milgrom and Pearce's (1991) formula. Then we introduce the turnover strategies, examine their equilibrium conditions, and compare with the first results. The extension to a model with sabotage is studied in Section 4.

2 Model

There is a team with n identical members (where $n \geq 2$), who are engaged in joint production over time. Each period, they have two alternatives; to work or to shirk. We denote their sets of period-actions by $\{W, S\}$, where W is to work and S is to shirk. It costs $c > 0$ for an agent to choose W, whereas S is costless. The outcome of the team production is binary, which is either good (success) or bad (failure), denoted by G and B, respectively. The outcome is stochastic, and its probabilities depend on the number of agents who choose W. We denote the probability of good outcome when k agents work by π_k. If the outcome is good, each agent receives a utility of $x > 0$. Failure gives no additional utility.

We assume that the outcome and the utility accrued from it are commonly observable among the agents, but are of undescribable nature. Namely, the outcome and payoff are unverifiable information, and therefore the team cannot commit to an explicit contract which specifies monetary transfer among them depending on the outcome. As a result, if k agents work in a given period, the expected period-payoff of each agent is $\pi_k x - c$ if he works, and $\pi_k x$ if he shirks. Thus, if we define the total benefit of success $X = nx$, the sum of expected period-payoffs when k agents work is $\pi_k X - kc$. We comment on this assumption of unverifiability in Section 3.

The following assumptions are made for the payoff and information structure of the period game.

Assumption 1 *For any $k \in \{1, 2, \cdots, n\}$, we assume:*

$$(\pi_k - \pi_{k-1})X > c > (\pi_k - \pi_{k-1})x, \tag{1}$$

$$\pi_k - \pi_{k-1} \quad \textit{is increasing in } k, \tag{2}$$

$$\pi_0 > 0, \quad \pi_n < 1. \tag{3}$$

(1) states that an agent's effort is always beneficial to the team, but it is not in the individual interests of the agent to choose W. Note that it implies monotonicity of π_k's. This assumption makes the period-game an n-person prisoners' dilemma, because full cooperation where all agents select W is efficient but S is the dominant action to all of them. (2) means that the incremental probability of success by an agent's effort is greater when more agents make efforts. In other words, the agents' efforts are complementary. This assumption is natural in the context of team production. One important consequences of this assumption is that it implies the monotone likelihood ratio property; namely, $(1 - \pi_{k-1})/(1 - \pi_k)$ is strictly increasing in k. Finally, (3) is an assumption of imperfect observations; both outcomes have a nonzero probability under any action pair.

One instance of parameters satisfying Assumption 1 is the following linear model;

$$\pi_k = \alpha k + \beta, \tag{4}$$

where $\alpha \in (c/X, c/x)$ and $n\alpha + \beta < 1$. In this model, the incremental probability of success by one agent's effort is independent of the others' actions.

In each period $t = 0, 1, 2, \cdots$, this period-game is played. A strategy of each agent in this infinitely repeated game is a mapping which determines a (randomized) action in each period, depending on what he observed in the past. We assume that the other agents' actions are not observable. We also assume that a public randomization device is available at the beginning of each period. Hence past observations are a collection of past outcomes, (a sequence of G or B) that agent's past actions (a sequence of W or S), and past realizations of the public randomization device. Therefore a strategy specifies actions in each period t as a function of a t-length sequence of signals, a t-length sequence of own actions, and a $(t + 1)$-length sequence of sunspots. A profile of strategies generates a sequence of expected period-payoffs, and we assume that each partner's overall utility from the strategy profile is the average discounted sum of the period payoffs. Formally, if a strategy profile $\sigma = (\sigma_i)_{i=1}^n$ generates a sequence of expected period payoffs $(u_i(t))_{t=0}^{\infty}$ for agent i, then his payoff of the repeated game is:

$$g_i(\sigma) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(t),$$

where $\delta \in (0, 1)$ is a common discount factor of the partners.

A strategy of agent i is *public* if it is independent of his own actions. A strategy profile is *public* if all agents play a public strategy. A strategy profile is a *public equilibrium* if it is a Nash equilibrium of the repeated game (in the standard sense), and if it is public. Because of the assumption of full support (3), any public history (a sequence of signals and sunspots) is on the path given *any* (possibly nonpublic) strategy profile. Hence the public equilibrium, sometimes simply called the equilibrium, satisfies the requirement of sequential rationality. Thus the public equilibrium is stronger than sequential equilibrium. We adopt the public equilibrium as our solution concept, mainly because the large part of literature also uses this solution concept (Radner, Myerson and Maskin (1986), Abreu, Pearce and Stacchetti (1990), Abreu, Milgrom and Pearce (1991) and Fudenberg, Levine and Maskin (1994)), and therefore it facilitates comparison with previous results.[3]

We call a public equilibrium *most efficient* or *second-best* if its payoff sum is no less than that of any public equilibrium of this model. While we have excluded a possibility of monetary transfer based on outcomes, we do not deny monetary transfers themselves. Therefore the welfare criterion in terms of the payoff sum is relevant. In what follows, we mainly study most efficient equilibria and their payoff sum when the agents are sufficiently patient.

[3]However, the restriction to public equilibrium is not warranted in terms of efficiency. Mailath, Matthews and Sekiguchi (2002) and Kandori and Obara (2006) provide examples where some nonpublic strategy sequential equilibria are more efficient than any public equilibrium. We do not know, however, whether our model is another example of that phenomenon.

3 Analysis

3.1 Equilibria with Initial Full Cooperation

Given the assumption that the agents are symmetric, one natural candidate for the second-best equilibrium is the one where all agents work in the initial period. In this subsection, we limit attention to a class of public strategy profiles which includes the above candidate, and derive a *uniform* bound on the sum of the agents' payoffs of the equilibria in this class, which applies to any level of discounting. This is a version of the uniform inefficiency result by Radner, Myerson and Maskin (1986), and the bound is characterized in the same way as the formula by Abreu, Milgrom and Pearce (1991).

A public strategy profile has *initial full cooperation* if the action profile $(W, W, \cdots, W)$ is played with a nonzero probability in period 0. Note that the definition of initial full cooperation is much weaker than the term suggests. Namely, the agents need not cooperate with probability 1. Moreover, it allows them to randomize in asymmetric ways. The grim trigger strategy profile has initial full cooperation.

Proposition 1 *Suppose there exists a most efficient equilibrium by a strategy profile with initial full cooperation. Then its payoff sum, denoted by V^*, satisfies*

$$V^* \leq X - n\frac{1-\pi_{n-1}}{\pi_n - \pi_{n-1}}c. \tag{5}$$

Proof. Let us fix a second-best equilibrium by a strategy profile with initial full cooperation, and $v = (v_i)_{i=1}^n$ be its payoff vector. Without loss of generality, we can assume that the equilibrium actions in period 0 do not depend on the sunspot of that period. Let

$$m = \big(\eta_1 W + (1-\eta_1)S, \eta_2 W + (1-\eta_2)S, \cdots, \eta_n W + (1-\eta_n)S\big)$$

be the mixed action profile in period 0 of this equilibrium. For each agent i, let $q_i^{a_i}$ be the probability of good outcome when agent i chooses $a_i \in \{W, S\}$ and all other agents follow m. Let $f_i(y)$ $(y \in \{G, B\})$ be agent i's expected continuation payoff from period 1 on, given that the signal in period 0 is y. Since the equilibrium has initial full cooperation, $\eta_i > 0$ for any i. Hence the following value equation and the incentive condition to choose W must hold for each agent i;

$$v_i = (1-\delta)\big(q_i^W x - c\big) + \delta\Big[q_i^W f_i(G) + \big(1-q_i^W\big) f_i(B)\Big] \tag{6}$$

$$\geq (1-\delta)\ q_i^S x + \delta\Big[q_i^S f_i(G) + \big(1-q_i^S\big) f_i(B)\Big]. \tag{7}$$

Note that (6) and (7) reduce to

$$(1-\delta)\Big\{c - \big(q_i^W - q_i^S\big)x\Big\} \leq \delta\big(q_i^W - q_i^S\big)\Big\{f_i(G) - f_i(B)\Big\}. \tag{8}$$

We claim that $\eta_i = 1$ must hold for each i. To show that, suppose on the contrary that $\eta_j < 1$ for some j. Then let us define the following new strategy profile. In period 0, all agents choose W independently of sunspots. From period 1 on, they conform to the original strategy profile. Let $(v_i')_{i=1}^n$ be the payoff vector of this new profile. Since the same $f_i(y)$'s specify the continuation payoffs, v_i''s are decomposed as follows, for each i.

$$v_i' = (1-\delta)(\pi_n x - c) + \delta\Big[\pi_n f_i(G) + \big(1-\pi_n\big) f_i(B)\Big]. \tag{9}$$

By (1), the left-hand-side of (8) is positive, and therefore we have $f_i(G) > f_i(B)$ for any i. Hence (2) and (8) imply:

$$(1-\delta)\Big\{c - \big(\pi_n - \pi_{n-1}\big)x\Big\} \leq \delta\big(\pi_n - \pi_{n-1}\big)\Big\{f_i(G) - f_i(B)\Big\} \tag{10}$$

for any i. By (10), any one-shot deviation from the new profile in period 0 does not pay. Since the continuation play from period 1 on forms a public equilibrium, this proves that the new profile is also an equilibrium.

Comparing (6) and (9), we see that the new profile has a strictly greater payoff sum than the original one, because (i) $\pi_n \geq q_i^S$ for any i and the inequality is strict for any $i \neq j$ (recall $\eta_j < 1$), and (ii) $f_i(G) > f_i(B)$ for any i. This is a contradiction against the second-best property of the original equilibrium, so that we must have $\eta_i = 1$ for any i. In other words, the agents cooperate with probability 1 in period 0.

Let us substitute (8) into (6) and then substitute $\eta_k = 1$ for all k. Then we have:

$$v_i \leq (1-\delta)(\pi_n x - c) + \delta f_i(G) - (1-\delta)(1-\pi_n)\frac{c - \big(\pi_n - \pi_{n-1}\big)x}{\pi_n - \pi_{n-1}}. \tag{11}$$

Since the equilibrium is most efficient, we have $\sum_{i=1}^n f_i(G) \leq \sum_{i=1}^n v_i$. Let us substitute it to the sum of (11) over i, and rearrange;

$$V^* = \sum_{i=1}^n v_i \leq \pi_n X - nc - n(1-\pi_n)\frac{c - \big(\pi_n - \pi_{n-1}\big)x}{\pi_n - \pi_{n-1}} = X - n\frac{1-\pi_{n-1}}{\pi_n - \pi_{n-1}}c,$$

which is equivalent to (5). Q.E.D.

Note that the upper bound on the sum of equilibrium payoffs given by Proposition 1 is independent of δ. Since (5) shows that the bound is less than the sum of the action profile $(W, W, \cdots, W)$, the difference is the size of the efficiency loss the team cannot avoid however patient they are. This is a restatement of the uniform inefficiency result by Radner, Myerson and Maskin (1986). To sustain a fully cooperative outcome in some period, the team must punish each agent in the future if a signal suggesting a deviation of that agent is realized. With only two signals, the bad signal must be used to punish *all* agents. This causes inevitable loss of efficiency for any equilibrium. Proposition 1 explicitly states the loss term, and it is a restatement of the formula presented in Abreu, Milgrom and Pearce (1991).

To what extent is the upper bound V^* tight? Note that it is possible that V^* is so small that the sum of the agents' minimax values of the period game exceeds it. Since the period-game is a prisoners' dilemma, its minimax value for each agent is his period-payoff of $(S, S, \cdots, S)$. Hence if $V^* \leq \pi_0 X$, no equilibrium with initial full cooperation exists under any δ. Otherwise, the payoff sum V^* is achieved by some trigger-type strategy profile if the agents are sufficiently patient. However, we are not much interested in when and how V^* is achieved, because we will examine whether there exist other types of public equilibria with a greater payoff sum than V^*.

3.2 Turnover Equilibrium

In Subsection 3.1, we have considered the strategy profiles in which all agents choose W with a positive probability in the initial period. This subsection examines quite a different class of strategies; at any history, there exists an agent who chooses S with certainty. In particular, we study a special class of those strategies.

Let us define the *turnover strategy profile with k shirkers*, where $k \in \{1, 2, \cdots, n-1\}$, as the following strategy profile, and let us denote it by τ^k and abbreviate it as the *k-TO strategy profile.* τ^k is described by an automaton. The state space is the set of all subsets of $N \equiv \{1, 2, \cdots, n\}$ with k elements. Formally, the state space Q is:

$$Q = \big\{N' \subseteq N : |N'| = k\big\}.$$

For example, if $n = 3$, then

$$Q = \begin{cases} \{\{1\}, \{2\}, \{3\}\} & \text{if } k = 1, \\ \{\{1,2\}, \{2,3\}, \{1,3\}\} & \text{if } k = 2. \end{cases}$$

The initial state is $\{1, 2, \cdots, k\}$. If the current period is in some state $N' \in Q$, then τ^k specifies that all agents in N' choose S and all other agents choose W.

The state transition rule is as follows. Suppose some $N' \in Q$ is the state of the current period. Then the state in the next period depends entirely on the outcome $y \in \{G, B\}$ in the current period. If $y = B$, then N' continues to be the state variable in the next period. If $y = G$, then we have two cases to consider. First, if $n - k \geq k$, then the state in the next period is selected from the set

$$\big\{N'' \in Q : N'' \cap N' = \emptyset\big\}$$

with equal probability. Second, if $n - k \leq k$, then the state in the next period is selected from the set

$$\big\{N'' \in Q : N \setminus N' \subseteq N''\big\}$$

with equal probability. Note that if $n - k = k$, the two rules are equivalent, because the above two sets are the same (in fact, equal to $\big\{N \setminus N'\big\}$).

The idea of this strategy profile is to allow exactly k agents to shirk at any history. The state variable corresponds to who is privileged to shirk, and the initial state is chosen arbitrarily; here the first k agents enjoy the privilege. If the current outcome is bad, then the same set of agents is given the privilege in the next period. If the current outcome is good, then a *turnover* occurs. That is, the set of agents who are allowed to shirk in the next period is chosen so that the maximum number of agents who are allowed to shirk lose the privilege in the next period. Hence if $n - k > k$, namely the case with more workers than shirkers, all current shirkers lose the privilege in the next period. The shirkers in the next period are chosen from the $n-k$ current workers, and the selection from the $n-k$ agents is made equiprobably. Instead, if $n-k<k$ and therefore there are less workers than shirkers, all current workers become shirkers in the next period. However, there are still $2k-n$ shirker slots, which are filled by the k current shirkers. Again, all possible selections from the k agents have equal probability. Finally, if $n-k=k$ and the number of workers and shirkers is the same, then the turnover occurs in a simplest form. That is, all agents change their roles in the next period.

The following result gives a necessary and sufficient condition for the k-TO profile τ^k to be a public equilibrium.

Proposition 2 *If*

$$\pi_{n-k}\big(\pi_{n-k}-\pi_{n-k-1}\big)X > \big\{(n-k)\pi_{n-k}+k\pi_{n-k-1}\big\}c \tag{12}$$

holds, then τ^k is a public equilibrium if and only if

$$\delta \geq \underline{\delta} \equiv \frac{L\big\{c-(\pi_{n-k}-\pi_{n-k-1})x\big\}}{\big(L-n\pi_{n-k}\big)\big\{c-(\pi_{n-k}-\pi_{n-k-1})x\big\}+\big(\pi_{n-k}-\pi_{n-k-1}\big)kc}, \tag{13}$$

where $L\equiv\max\{n-k,k\}$. If (12) does not hold, then τ^k is not a public equilibrium under any δ.

Proof. In τ^k, the play is stationary and the agents are symmetric in terms of the transition probabilities between workers and shirkers. Hence let v be the payoff of τ^k for each agent $i\in\{1,2,\cdots,k\}$ and let w be the payoff of τ^k for each agent $i\in\{k+1,\cdots,n\}$. Then we have the following value equations:

$$v=(1-\delta)\pi_{n-k}x+\delta\left[\pi_{n-k}\frac{n-k}{L}w+\Big(1-\pi_{n-k}\frac{n-k}{L}\Big)v\right], \tag{14}$$

$$w=(1-\delta)\big(\pi_{n-k}x-c\big)+\delta\left[\pi_{n-k}\frac{k}{L}v+\Big(1-\pi_{n-k}\frac{k}{L}\Big)w\right]. \tag{15}$$

From (14) and (15), we obtain

$$v-w=\frac{(1-\delta)Lc}{(1-\delta)L+\delta n\pi_{n-k}}. \tag{16}$$

Note that (16) implies $v>w$.

Suppose that the current period is in some state $N'\in Q$. Then the agents in N' has no incentive to make a one-shot deviation; a deviation to W reduces his period-payoff due to (1), and it makes the unfavorable continuation payoff w more likely. Next, let us consider a one-shot deviation by an agent not in N'. By symmetry, this is the only relevant incentive problem, and therefore τ^k is a public equilibrium if and only if this one-shot deviation does not pay.

The one-shot deviation does not pay if and only if

$$w \geq (1-\delta)\pi_{n-k-1}x + \delta\left[\pi_{n-k-1}\frac{k}{L}v + \left(1 - \pi_{n-k-1}\frac{k}{L}\right)w\right],$$

$$\therefore\ (1-\delta)\Big\{c - (\pi_{n-k} - \pi_{n-k-1})x\Big\} \leq \delta(\pi_{n-k} - \pi_{n-k-1})\frac{k}{L}(v-w). \tag{17}$$

Let us substitute (16) into (17), and divide by $1-\delta$;

$$c - (\pi_{n-k} - \pi_{n-k-1})x \leq \delta(\pi_{n-k} - \pi_{n-k-1})\frac{kc}{(1-\delta)L + \delta n \pi_{n-k}}.$$

After rearranging, we have;

$$\begin{aligned} &\delta\Big[\big(L - n\pi_{n-k}\big)\big\{c - (\pi_{n-k} - \pi_{n-k-1})x\big\} + \big(\pi_{n-k} - \pi_{n-k-1}\big)kc\Big] \\ &\geq L\big\{c - (\pi_{n-k} - \pi_{n-k-1})x\big\}. \end{aligned} \tag{18}$$

(12) is equivalent to the strict inequality version of (18) evaluated at $\delta = 1$. Therefore if (12) holds, the coefficient of δ in (18) is strictly positive. Hence (18) is equivalent to (13), which proves the first part of the proposition.

Next, suppose (18) holds for some $\delta \in (0,1)$. Then the coefficient of δ in (18) is strictly positive. Thus the strict inequality version of (18) holds at $\delta = 1$, which is (12). Hence (12) is a necessary condition for τ^k to be a public equilibrium for some δ. Taking a contraposition, we prove the second part. Q.E.D.

Proposition 2 states that τ^k is a public equilibrium for patient agents, if X is large or if π_{n-k-1} is small in comparison with other parameters. Under τ^k, an agent obviously benefits from starting a period as a shirker, rather than a worker, for any level of discounting. This creates a rent for a shirker. It provides an incentive to work if he is assigned to work in the current period, because W increases a probability of good outcome and therefore that of turnover. However, the size of the rent is already fixed and depends entirely on c, k and π_k. For an equilibrium, it must be large enough to provide incentives, which is the case if the cost associated with a current effort is small (large X), or if shirking reduces the probability of success to a great extent (small π_{n-k-1}). Additionally, he of course must be patient enough.

In general, it is not obvious how the set of k satisfying (12) looks like. One special case where the characterization of the set is easy is the linear technology which satisfies (4). In this case, (12) reduces to

$$\alpha X > \left\{n - \frac{\alpha k}{\alpha(n-k) + \beta}\right\}c,$$

whose right-hand-side is strictly decreasing in k. Hence if (12) is satisfied for some k, it is satisfied for any $k' > k$. In other words, there exists $\hat{k}$ such that the set of k satisfying (12) has the form $\{\hat{k}, \hat{k}+1, \cdots, n-1\}$ if (4) holds.[4]

The next question to ask is whether the turnover equilibria with some number

[4]However, it is possible that $\hat{k} = n$ and therefore the set is empty.

of shirkers can be a candidate for the second-best equilibrium. The next subsection answers to that question.

3.3 Optimality of Shirking

Proposition 2 in the previous subsection is most interesting when it holds for $k = 1$; the case where a single agent is allowed to shirk. First, the 1-TO equilibrium is most efficient among all turnover equilibria. Second, and more importantly, it is a second-best equilibrium if it can be shown to be more efficient than any public equilibrium with initial full cooperation.

Proposition 3 *If*

$$\pi_{n-1}\big(\pi_{n-1} - \pi_{n-2}\big)X > \big\{(n-1)\pi_{n-1} + \pi_{n-2}\big\}c, \tag{19}$$

$$\delta \geq \frac{(n-1)\big\{c - (\pi_{n-1} - \pi_{n-2})x\big\}}{\big\{n(1-\pi_{n-1}) - 1\big\}\big\{c - (\pi_{n-1} - \pi_{n-2})x\big\} + \big(\pi_{n-1} - \pi_{n-2}\big)c}, \tag{20}$$

$$\pi_{n-1}X - (n-1)c > X - n\frac{1 - \pi_{n-1}}{\pi_{n-1} - \pi_{n-2}}c \tag{21}$$

hold, then τ^1 is a second-best public equilibrium. Moreover, any second-best public equilibrium has the property that at any history exactly one agent is prescribed to shirk given the history.

Proof. By Proposition 2, τ^1 is a public equilibrium because (19) and (20) hold. Since the payoff sum of τ^1 is $\pi_{n-1}X - (n-1)c$, (21) and Proposition 1 imply that τ^1 is more efficient than any public equilibrium with initial full cooperation.

Fix a most efficient public equilibrium, and let $(v_i)_{i=1}^n$ be its payoff vector. Also, let $(u_i)_{i=1}^n$ be its expected period-payoff vector in period 0, and let $\big(f_i(y)\big)_{i=1}^n$ $(y \in \{G, B\})$ be its continuation equilibrium payoff vector from period 1 on if the signal in period 0 is y. Then we have the following value equation for the payoff sum:

$$\sum_{i=1}^n v_i = (1-\delta)\sum_{i=1}^n u_i + \delta\Big\{\hat{\pi}\sum_{i=1}^n f_i(G) + (1-\hat{\pi})\sum_{i=1}^n f_i(B)\Big\}, \tag{22}$$

where $\hat{\pi}$ is the probability of G in period 0.

Since the second-best equilibrium has no initial full cooperation,

$$\sum_{i=1}^n u_i \leq \pi_{n-1}X - (n-1)c \leq \sum_{i=1}^n v_i, \tag{23}$$

where the second inequality follows because τ^1 is an equilibrium. We also have $\sum_{i=1}^n v_i \geq \sum_{i=1}^n f_i(y)$ for each $y \in \{G, B\}$. Substituting this and (23) into (22), we find

$$\sum_{i=1}^n v_i = \sum_{i=1}^n u_i = \pi_{n-1}X - (n-1)c = \sum_{i=1}^n f_i(y) \tag{24}$$

for any $y \in \{G, B\}$. This implies that τ^1 is most efficient. Also (24) implies that the expected period-payoff sum of any second-best equilibrium in period 0 is $\pi_{n-1}X - (n-1)c$. Since the equilibrium does not have initial full cooperation, exactly one agent shirks in period 0 of this equilibrium. (24) also implies that any continuation equilibrium of a most efficient equilibrium is also most efficient. Therefore, any second-best equilibrium must specify exactly one agent to shirk at any history, which completes the proof. Q.E.D.

The first part of Proposition 3 implies that the 1-TO equilibrium is second-best for patient partners if the efficiency loss associated with equilibria with initial full cooperation is large and if the 1-TO strategy profile is indeed a public equilibrium. The problem is whether the efficiency loss from the uniform inefficiency result is greater than the efficiency loss from letting one agent shirk. Roughly speaking, the former loss tends to be greater than the latter if the success is a rare event even if all agents work. If the success is unlikely, failure does not provide much information as to whether someone deviates from a fully cooperative profile. Therefore the incentive costs to let all agent work tend to be large. One set of parameters satisfying the conditions of Proposition 3 is: $n = 4$, $\pi_i = 0.03(i+1)$, $c = 2$ and $x = 63$ (hence $X = 252$). In this case, the 1-TO equilibrium is most efficient if $\delta \geq 275/281$.

The second part of Proposition 3 implies that, if the conditions for this proposition are met, the most efficient equilibrium *always* has the form of shifting the status as a single shirker depending on past outcomes. Thus the second-best requires presence of a slacker in the team. Indeed, the slacker plays a similar role to the principal in the context of moral hazard in teams (Holmstrom (1982)). The principal in those models of teams does not perform something productive to the team. He is simply a budget-breaker and pays to the agents if their performance is good. In our model of repeated games, the slacker does not pay to the other agents directly. He rather pays out his future rent, by giving up his privilege via turnover.[5]

While we are interested in the issue of optimality of shirking, we have so far only considered a special type of strategies; the 1-TO strategy profile. It may be that other profiles sustain equally efficient outcomes under a wider range of parameters. The following result proves that this is not the case. The statement needs more definitions. A public strategy profile is *k-shirker type* ($k \in \{1, 2, \cdots, n-1\}$) if at any history the profile prescribes some k agents to choose S and the remaining $n-k$ agents to select W. An equilibrium is *k-shirker type* if it is a k-shirker type strategy profile.

Proposition 4 *If a 1-shirker type strategy profile σ is a public equilibrium, then* (19) *and* (20) *must hold.*

Proof. Let E be the set of 1-shirker type equilibrium payoffs of some agent i. By symmetry, E does not depend on the choice of i. Let $w = \min E$, and let σ be a 1-shirker type equilibrium whose payoff of agent 1 is w. For each agent i, let v_i, $f_i(G)$ and $f_i(B)$ be agent i's equilibrium payoff and continuation payoffs from period 1 on given that the signal in period 0 is G or B, respectively.

[5] Rayo (2007) studies a model of repeated team production with individual signals, and examines how a principal is determined endogenously. Our model is different in the sense that we rather study endogenous dynamics of the status of a principal.

We first show that σ prescribes agent 1 to choose W in period 0. Suppose otherwise. Then agent 1 has the following value equation:

$$w = (1-\delta)\pi_{n-1}x + \delta\big\{\pi_{n-1}f_1(G) + (1-\pi_{n-1})f_1(B)\big\}.$$

Note that the continuation equilibrium of σ is also 1-shirker type. Hence $f_1(G) \in E$ and $f_1(B) \in E$, and by the definition of w, we obtain:

$$w \geq (1-\delta)\pi_{n-1}x + \delta w.$$

Hence $w \geq \pi_{n-1}x$. Since $v_i \geq w$ for any i, we must have $\sum_{i=1}^{n} v_i \geq \pi_{n-1}X$. This is a contradiction, because any 1-shirker type strategy profile has the payoff sum of $\pi_{n-1}X - (n-1)c$. As a result, agent 1 chooses W in period 0.

Without loss of generality, we can assume that it is agent n who is prescribed to shirk in period 0. Then the one-shot deviation by any agent $i \leq n-1$ in period 0 does not pay:

$$(1-\delta)\big\{c - (\pi_{n-1} - \pi_{n-2})x\big\} \leq \delta(\pi_{n-1} - \pi_{n-2})\big\{f_i(G) - f_i(B)\big\}. \tag{25}$$

Since $\sum_{i=1}^{n} f_i(G) = \sum_{i=1}^{n} f_i(B)$, (25) implies

$$\begin{aligned}\delta(\pi_{n-1} - \pi_{n-2})\big\{f_n(B) - f_n(G)\big\} &= \delta(\pi_{n-1} - \pi_{n-2})\sum_{i=1}^{n-1}\big\{f_i(G) - f_i(B)\big\} \\ &\geq (1-\delta)(n-1)\big\{c - (\pi_{n-1} - \pi_{n-2})x\big\}. \end{aligned}\tag{26}$$

Since $\sum_{i=1}^{n} f_i(B) = \pi_{n-1}X - (n-1)c$ and $f_i(y) \geq w$ for any i and any $y \in \{G, B\}$, it follows that

$$\begin{aligned} f_n(B) - f_n(G) &= \pi_{n-1}X - (n-1)c - \sum_{i=1}^{n-1} f_i(B) - f_n(G) \\ &\leq \pi_{n-1}X - (n-1)c - nw. \end{aligned}\tag{27}$$

From the value equation of agent 1, we have

$$\begin{aligned} w =& (1-\delta)\big(\pi_{n-1}x - c\big) + \delta\big\{\pi_{n-1}f_1(G) + (1-\pi_{n-1})f_1(B)\big\} \\ \geq& \pi_{n-1}x - c + \frac{\delta}{1-\delta}\pi_{n-1}\big\{f_1(G) - f_1(B)\big\} \\ \geq& \frac{\pi_{n-2}}{\pi_{n-1} - \pi_{n-2}}c, \end{aligned}\tag{28}$$

where the first inequality follows from $f_1(B) \geq w$, and the second one from (25).

Let us substitute (28) into (27), and then substitute it into (26). Then we obtain:

$$\begin{aligned} &\delta(\pi_{n-1} - \pi_{n-2})\Big\{\pi_{n-1}X - \frac{(n-1)\pi_{n-1} + \pi_{n-2}}{\pi_{n-1} - \pi_{n-2}}c\Big\} \\ &\geq (1-\delta)(n-1)\big\{c - (\pi_{n-1} - \pi_{n-2})x\big\}. \end{aligned}\tag{29}$$

Since each side of (29) must be positive, (19) must hold. In this case, (29) is equivalent to (20). Q.E.D.

Proposition 4 reveals that the 1-TO strategy profile is a representative of the 1-shirker type strategy profiles. It implies that even if (21) is satisfied, the second-best equilibrium payoff sum is necessarily less than $\pi_{n-1}X - (n-1)c$ if either (19) or (20) is violated. As far as optimality of the 1-shirker type strategy profiles is concerned, no generality is lost when we limit attention to τ^1.

We conclude this section by pointing out the role of the assumption that the outcome is not verifiable. If it were verifiable, then the team can write a contract specifying monetary transfers among its members depending on the outcome. Suppose (19) holds, and consider the following static contract: if the outcome is G, then agent n pays $(x - \varepsilon)/(n-1)$ to each of the other $n-1$ agents, where $\varepsilon > 0$ satisfies

$$(X - \varepsilon)(\pi_{n-1} - \pi_{n-2}) > (n-1)c. \tag{30}$$

Note that (19) guarantees existence of such $\varepsilon > 0$. If the outcome is B, no monetary transfer is taken place.

Given this contract, it forms a static equilibrium for all agents $i \leq n-1$ to choose W and for agent n to choose S. Indeed, (30) implies that it is optimal for agent $i \leq n-1$ to choose W given the other agents' actions. Thus the same payoff sum $\pi_{n-1}X - (n-1)c$ is sustained as a repetition of one-period contractual outcomes. In other words, our analysis demonstrates that lack of verifiability is sometimes resolved by repeated play among the agents.

4 Discussions

So far we have assumed that the agents have a binary choice between a positive effort and a zero effort each period. In this section, we study an extended model where they have a *negative effort* as a third option. This action is costly and reduces the probability of good outcome given the others' actions, and therefore it is a dominated action in the period game. Nevertheless it sometimes makes much sense to have this type of options in repeated games, because it changes the minimax values of players and therefore the equilibrium payoffs. This option is also relevant in terms of the turnover equilibria, because the agents with privilege may have an incentive to exercise the option in order to avoid turnover.

Formally, the action set of each agent in each period is now $\{W, S, D\}$, where D means destructive activity. The set of signals and the cost of W remains the same; $\{G, B\}$ and $c > 0$. Let $d > 0$ be the cost of choosing D. For k and l such that $0 \leq k + l \leq n$, let $\pi_{k,l}$ be the probability of G when k agents work and l agents choose D. We assume for any k and l such that $0 \leq k + l < n$, $\pi_{k+1,l} > \pi_{k,l} > \pi_{k,l+1}$. We define $\pi_k \equiv \pi_{k,0}$, and assume that those π_i's satisfy Assumption 1.[6]

Also in this extended model, we can define the turnover strategy profile in the same way. The next result limits attention to τ^1 and states that the equilibrium conditions

[6] We can think of other reasonable assumptions on $\pi_{k,l}$'s, but they are unnecessary here.

are the same.

Proposition 5 τ^1 *is a public equilibrium if and only if* (19) *and* (20) *hold.*

Proof. Since the introduction of action D does not change the definition of τ^1, the payoffs of agents 1 and 2, denoted by v and w respectively, are decomposed as follows.

$$v = (1-\delta)\pi_{n-1}x + \delta\big\{\pi_{n-1}w + (1-\pi_{n-1})v\big\},$$
$$w = (1-\delta)(\pi_{n-1}x - c) + \delta\left\{\frac{\pi_{n-1}}{n-1}v + \Big(1 - \frac{\pi_{n-1}}{n-1}\Big)w\right\},$$

from which we obtain

$$v - w = \frac{(1-\delta)(n-1)c}{(1-\delta)(n-1) + \delta n\pi_{n-1}}. \tag{31}$$

As before, in τ^1, the agent prescribed to choose S has no incentive to choose W. Also the agent prescribed to choose W has no incentive to choose S if and only if (19) and (20) hold. He has no incentive to choose D, since it is more costly and reduces the probability of G more than S. So suppose (19) and (20) hold, and let us consider the one-shot deviation by a shirker such that he chooses D and then conforms to τ^1.

The gain from this deviation is:

$$D \equiv (1-\delta)(-d + \pi_{n-1,1}x) + \delta\big\{\pi_{n-1,1}w + (1-\pi_{n-1,1})v\big\} - v$$

Substituting (31) into this yields:

$$D = (1-\delta)\left[-d - (\pi_{n-1} - \pi_{n-1,1})\frac{\big\{(1-\delta)(n-1) + \delta n\pi_{n-1}\big\}x - \delta(n-1)c}{(1-\delta)(n-1) + \delta n\pi_{n-1}}\right].$$

Note that (19) implies $\pi_{n-1}X > (n-1)c$. Therefore,

$$\begin{aligned}&\big\{(1-\delta)(n-1) + \delta n\pi_{n-1}\big\}x - \delta(n-1)c\\ =&(1-\delta)(n-1)x + \delta\big\{\pi_{n-1}X - (n-1)c\big\} > 0\end{aligned}$$

follows. This proves that $D < 0$, and the necessary and sufficient condition for τ^1 to be a public equilibrium is (19) and (20), as is desired. Q.E.D.

The option to sabotage production is irrelevant simply because it is too costly for an agent with privilege to choose it. It is not only costly per se, but also reduces benefits from current success. Lazear (1989) lays out the idea that the agents may engage in detrimental activities if they are evaluated based on relative performance. Tournaments are a typical example, and Lazear (1989) points out importance of the compressed wage schedule. In our model the payments must be made from transfer of future payoffs, and the range of possible transfers is limited.[7] The wages are thus already compressed, and the equilibrium condition in our original model, virtually

[7] Even if they are sufficiently patient, the benefit from turnover is not long-lived and therefore the size of incentives provided via turnover is not so large.

stating that these compressed wages still provide incentives, also provides incentives not to choose D.

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