

# All-Pay Auctions with Private Values and Resale <sup>\*</sup>

Qiang Gong

National School of Development

Peking University

Yong Sui<sup>†</sup>

School of Economics

Shanghai Jiao Tong University

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## Abstract

This paper studies all-pay auctions with resale opportunities within an independent-private-value framework. Given the existence of a resale market, the primary players will compete more aggressively over an indivisible prize. We characterize a symmetric equilibrium for all-pay auctions with private values and resale and derive a revenue-ranking result for all-pay auctions with and without resale opportunities. Depending on the first-stage winner's ability to extract surplus in the resale stage, the initial seller may or may not benefit from excluding some players from the primary competition thus creating a resale market.

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<sup>†</sup>The corresponding author: 535 Fahuazhen Rd. Shanghai Jiao Tong University, Shanghai, China 200052.  
Email: ysui@sjtu.edu.cn

# 1 Introduction

This paper studies all-pay auctions with private values and resale opportunities. In contrast to standard auctions where only winners are required to make payments, all-pay auctions exhibit special characteristic of unconditional payment, that is, bidders always pay their bids regardless winning or losing. All-pay auctions or equivalent models have been widely used to model a variety of economic and social instances of conflict and competition such as lobbying, contests and tournaments, political campaigns, patent races, and so on.<sup>1</sup> In practice, there are many instances a static all-pay auction model could not account for. Participants in such instances often face aftermarket competitions.

Consider an all-pay auction followed by a resale mechanism. The motivating example is R&D contests. Very often the winners of those contests do not consume the prize themselves; instead, they would like to sell the prizes on the secondary market. Moreover, winners of patent races usually sell the patents to others in order to pursue more profits. Forseeing the resale opportunity certainly will change those first-stage competitors' strategic behaviors. A static all-pay auction model is not rich enough to study such issues, which leads us to consider an all-pay auction with resale.

There could be many forms of the resale mechanism. It could be an auction, a fixed-price selling mechanism or even a multi-lateral bargaining process, and so on. In this paper, we study two formats of the resale mechanism: a second-price auction and an optimal auction as Myerson (1981) characterizes. When the seller on the secondary market conducts a second-price auction to resell the prize, her private value serves as a natural reserve price. In the case of an optimal auction, the seller will optimally choose a reserve price in order to maximize her expected surplus from resale. Depending on the exogenously-chosen resale mechanism, the first-stage players behave differently. This is the main theme we will investigate in this paper.

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<sup>1</sup>See, for example, Baye et al. (1993), Krishna and Morgan (1997), and Moldovanu and Sela (2001).

To control the information flow between these two stages, we assume that the winning bid will be announced after the first-stage all-pay auction. The aftermarket buyers observe this information, infer the winner's private value, and compete over the prize either in a second-price auction or in an optimal auction. Since the seller attaches a positive value to the prize, she will not resell it unless she extracts a nonnegative surplus. Hence, if the second-stage competition takes place through a second-price auction, all potential buyers take the seller's value as a reserve price. Given this, bidding his or her private valuation remains a weakly dominant strategy in the second-price auction. Foreseeing this connection between the expected resale price and the bids submitted in the first-stage all-pay auction, the first-stage bidders have signalling incentives. Thus, their valuations will be endogenously determined due to resale opportunities.<sup>2</sup> On the contrary, if the second-stage competition takes place through an optimal auction, the seller will optimally determine the reserve price and make it publicly known to all aftermarket buyers. There is no signalling incentive since the seller will optimally determine the reserve price based on her true valuation.

Given existence of a resale market, the potential gain makes the primary players compete more aggressively over an indivisible prize. We characterize a symmetric equilibrium for all-pay auctions with independent private values and resale. We derive a revenue-ranking result for all-pay auctions with and without resale opportunities. If the resale is conducted through a second-price auction, we obtain a revenue equivalent result: the expected revenue to the initial seller remain the same whether there is market separation or not. If the resale is conducted through an optimal auction, the initial seller never gains and may lose in terms of expected revenue. Given those two resale mechanisms, the initial seller does not gain anything in terms of expected revenue. However, depending on the first-stage winner's ability of extracting surplus in the resale stage, the initial seller may benefit from excluding some players from the primary competition thus creating a resale market.

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<sup>2</sup>Similar results could be found in Goeree (2003), Haile (2003) and Sui (2007).

Krishna and Morgan (1997) study all-pay auctions using a general symmetric model. They characterize equilibrium strategies for both first-price and second-price all-pay auctions and derive a revenue ranking results. Sui (2007) extends Krishna and Morgan (1997)'s general symmetric setting to incorporate resale possibilities into all-pay auctions. He characterizes symmetric bidding equilibria for both first- and second-price all-pay auctions with resale. Based on these equilibria, he compares the two formats from the perspective of a revenue-maximizing seller.<sup>3</sup>

This paper is also related to the literature regarding auctions with resale. Bikhchandani and Huang (1989) present a model with symmetric information applicable to treasury bill auctions, where pure common values and a competitive resale market are assumed. Resale takes place because most bidders in the first stage are speculators and bid for resale. They characterize equilibrium bidding strategies for both discriminatory and uniform-price auctions. Provided existence of symmetric equilibria, they show that uniform-price auctions generate no less expected revenue than discriminatory auctions. Using a model with independent private values, Haile (2003) studies auctions with resale under private uncertainties. Resale takes place because of the discrepancy between the estimated values at the time of bidding and the true values realized after the auction. He characterizes equilibrium bidding strategies for first-price, second-price and English auctions followed by resale which could be formalized as an optimal auction or an English auction. He argues that the option to resell creates endogenous valuations and induces signaling incentives that may revert the revenue results obtained in the literature that assumes no resale. In our model, resale takes place because of new market entrants, which differs our paper from most of the literature.

In the following section, we present the main model and investigate two resale mechanisms. Section 3 contains a detailed example to show our main results. Section 4 concludes.

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<sup>3</sup>Moreover, Sui (2007) shows that the revenue ranking result derived in Krishna and Morgan (1997) remains even if we introduce resale opportunities to all-pay auctions.

## 2 The Model

The two-stage game proceeds as follows. In the first stage,  $N$  risk-neutral players compete over an indivisible prize through an all-pay auction. Each first-stage player has independent private value drawn from an identical distribution  $F$  with density  $f$  over support  $[0, \omega]$ . The player who submits the highest bid wins the auction, and all players pay their bids. In the second stage, the winner could resell the prize through a resale mechanism. In this study, we consider two kinds of resale mechanisms: a second-price auction or an optimal auction. There are  $M$  potential buyers in the second stage and we assume that each second-stage buyer has independent private value drawn from the same distribution  $F$  with the same support. Moreover, we assume that the first-stage losing players do not enter the resale market, hence there exists a market separation.<sup>4</sup> As for the information flow between these two stages, we assume that the winning bid will be announced after the first stage, hence the secondary players could use those announced winning bids to infer the first-stage winner's private valuation about the prize.

### 2.1 Second-price Resale Mechanism

We solve the game through backward induction. We start from the second stage. Suppose there exists a monotonic increasing equilibrium bidding strategy  $\beta$  for the first-stage players. If the announced winning bid is  $\bar{b}$ , the second-stage buyers infer that the winning bidder's private valuation  $r = \beta^{-1}(\bar{b})$ . Hence, in the second stage,  $r$  will serve as the reserve price in the second-price auction. Individual rationality implies that the first-stage winner will keep the prize if no one bids higher than  $r$ .

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<sup>4</sup>This is important to our results. If the losing bidders enter the secondary market, they may strategically lose the first stage and then try to win the prize in the second stage. See Pagnozzi (2007).

The expected payment of a buyer with value  $v \geq r$  is given by

$$m(x, r) = rG(r) + \int_r^x yg(y)dy \quad (1)$$

where  $G$  is the distribution of  $y \equiv \max_{j \neq 1} x_j$  with a density  $g$ . Here  $G(r) = F(r)^{M-1}$ . Then the ex ante expected payment of a buyer is now

$$\begin{aligned} E[m(x, r)] &= \int_r^\omega m(x, r)f(x)dx \\ &= r(1 - F(r))G(r) + \int_r^\omega y(1 - F(y))g(y)dy \end{aligned}$$

Suppose the winner's private value is  $x$  and bids as if her true value is  $r$ , the overall expected payoff of the first-stage winner from resale is

$$\Pi_0 = M \times E[m(x, r)] + F(r)^M x \quad (2)$$

Therefore, in the first stage, suppose all other bidders follow  $\beta$ , a bidder learns his private value  $x$ , and submits  $b$  to maximize

$$H(\beta^{-1}(b))\Pi_0 - b \quad (3)$$

where  $H$  is the distribution of the highest private value among a first-stage bidder's competitors. Maximizing (3) over  $b$ , the resulting first-order condition yields:

$$\begin{aligned} \beta'(x) &= MF(x)^{M+N-2} - MF(x)^{M+N-1} \\ &+ M(N-1)xF(x)^{M+N-3}f(x) - (N-1)(M-1)xF(x)^{M+N-2}f(x) \\ &+ M(N-1)F(x)^{N-2}f(x) \int_x^\omega y(1 - F(y))g(y)dy \end{aligned}$$

Given the boundary condition  $\beta(0) = 0$ , the symmetric equilibrium is

$$\begin{aligned}\beta(x) &= \int_0^x MF(t)^{M+N-2} dt - \int_0^x MF(t)^{M+N-1} dt \\ &+ \int_0^x M(N-1)tF(t)^{M+N-3} f(t) dt - \int_0^x (N-1)(M-1)tF(t)^{M+N-2} f(t) dt \\ &+ \int_0^x M(N-1)F(t)^{N-2} f(t) \int_t^\omega y(1-F(y))g(y) dy dt\end{aligned}$$

By manipulation we could simplify the bidding strategy<sup>5</sup>:

$$\begin{aligned}\beta(x) &= MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y) dy - \int_0^x F(t)^{M+N-1} dt \\ &+ Mx F(x)^{M+N-2} - (M-1)x F(x)^{M+N-1}\end{aligned}$$

It is trivial to verify that  $\beta(\cdot)$  is monotonic increasing, differentiable, and hence is indeed an equilibrium for the two-stage game. Hence, we have the following proposition.

**Proposition 1.** *In an all-pay auction with private values and resale, if the resale is conducted through a second-price auction, the symmetric equilibrium strategy for the first-stage players is given by  $\beta$ .*

The expected revenue to the initial seller in an all-pay auction with resale is given by

$$R = N \int_0^\omega \beta(x) f(x) dx \quad (4)$$

In an all-pay auction with  $N$  bidders and without resale, the equilibrium strategy is given by

$$\tilde{\beta}(x) = \int_0^x y \tilde{g}(y) dy \quad (5)$$

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<sup>5</sup>See Appendix B for details.

where  $\tilde{g}$  is the density function of  $\tilde{G} = F(\cdot)^{N-1}$ , and the expected revenue is

$$\tilde{R} = N \int_0^\omega \tilde{\beta}(x) f(x) dx = N \int_0^\omega y(1 - F(y)) \tilde{g}(y) dy \quad (6)$$

In an all-pay auction with  $M + N$  bidders and without resale, the equilibrium strategy is given

$$\hat{\beta}(x) = \int_0^x y \hat{g}(y) dy \quad (7)$$

where  $\hat{g}$  is the density function of  $\hat{G} = F(\cdot)^{M+N-1}$ , and the expected revenue is

$$\hat{R} = (M + N) \int_0^\omega \hat{\beta}(x) f(x) dx = (M + N) \int_0^\omega y(1 - F(y)) \hat{g}(y) dy \quad (8)$$

Clearly we have  $\hat{R} \geq \tilde{R}$  since

$$\hat{R} = E[Y_2^{(M+N)}] \geq E[Y_2^{(N)}] = \tilde{R} \quad (9)$$

where  $Y_2^{(M+N)}$  is the second highest of  $M + N$  values. Similarly,  $Y_2^{(N)}$  is the second highest of  $N$  values. For the general revenue ranking, we have the following result.

**Theorem 1.** *In an all-pay auction with or without resale, the expected revenue for the initial seller satisfies*

$$R = \hat{R} \geq \tilde{R} \quad (10)$$

*Proof.* See Appendix. □

**Remark 1.** *Without resale, the initial seller could benefit from bringing more competitors to an all-pay auction. With resale opportunities, if the resale is conducted through a second-price auction, the initial seller does not lose anything in terms of expected revenue due to market separation. In particular, if the initial seller could bring all those aftermarket buyers to the first-stage competition, she will obtain the same expected revenue.*



If we think of the initial seller as a social planner, she may care about social efficiency besides allocative efficiency and wish to reduce the total expenditure in the first stage.<sup>6</sup> In this case, the social planner could reduce the total expenditure by allowing less players into competition and forbidding resale. In practise, there are many reasons to limit entry into the first-stage competition. For instances, in research contests or tournaments for new technology, entry into tournaments is limited to save on the cost of conducting and evaluating the competition, and to minimize the duplication of fixed costs. At times entry into tournaments is limited to raise the level of contestant effort. Fullerton and McAfee (1999) investigate the entry-limiting problem and show that, for a large class of contests the optimal number of competitors is two. They further propose to use a variant of the all-pay auction to efficiently select the most qualified contestants to compete.

## 2.2 Optimal Resale Auction

Now suppose the first-stage winner could set a reserve price to extract more surplus. For simplicity, we assume that he uses the optimal auction characterized by Myerson (1981). In our case, the first-stage winner resells the prize through an optimal auction mechanism. Moreover, from Myerson's characterization, we know that a second-price auction with an optimally-set reserve price is indeed an optimal auction.

As in Myerson (1981), we define the virtual valuation function as:

$$\psi(x) \equiv x - \frac{1 - F(x)}{f(x)} \quad (11)$$

We focus on the regular case, i.e., the virtual value function  $\psi(\cdot)$  is an increasing function of

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<sup>6</sup>For example, if the government decides to allocate a monopolistic license, it will induce firms to lobby for the rent. Of course, other interested firm could obtain the licence through taking over the winning firm in the second stage.

the true value  $x$ . Since

$$\Psi(x) \equiv x - \frac{1}{\lambda(x)} \quad (12)$$

where  $\lambda \equiv f/(1 - F)$  is the hazard rate function associated with  $F$ , a sufficient condition for regularity is that  $\lambda(\cdot)$  is increasing. We will assume that the design problem faced for the first-stage winner is regular in what follows.

In an optimal auction, the seller will set the reserve price  $r^* = \Psi^{-1}(x)$ , where  $x$  is the true valuation of the first-stage winner.

Substitute  $r^*$  with  $r$ , we could derive the continuation payoff for the first-stage winner:

$$\Pi_0^* = M \times E[m(x, r^*)] + F(r^*)^M x \quad (13)$$

where

$$E[m(x, r^*)] = r^*(1 - F(r^*))G(r^*) + \int_{r^*}^{\omega} y(1 - F(y))g(y)dy \quad (14)$$

Therefore, in the first stage, suppose all other bidders follow  $\beta_o$ , a bidder learns his private value  $x$ , and submits  $b$  to maximize

$$H(\beta_o^{-1}(b))\Pi_0^* - b \quad (15)$$

Maximizing (15) over  $b$ , the resulting first-order condition yields:

$$\beta_o'(x) = (N - 1)F(x)^{N-2}f(x)\Pi_0^* \quad (16)$$

and we have  $r^* = \Psi^{-1}(x)$ . Since  $\beta_o(0) = 0$ , we have

$$\beta_o(x) = \int_0^x (N - 1)F(t)^{N-2}f(t)\Pi_0^*(r^*(t))dt \quad (17)$$

**Remark 2.** *The equilibrium bidding strategy  $\beta_o$  exhibits a much simpler expression than  $\beta$ , the equilibrium strategy we characterize when the first-stage winner uses a plain second-price auction to resell the prize. Certainly different resale mechanisms have different impact on those first-stage bidders' strategies. With second-price resale mechanism, the first-stage bidders have incentives to signal their private values to aftermarket buyers.<sup>7</sup> With optimal resale mechanism, the first-stage bidders have no such incentives since the winner will optimally set the reserve price according to her true value instead of her signalled value.*

Again, we could easily verify that  $\beta_o(\cdot)$  is monotonic increasing, differentiable and continuous, and thus is indeed an equilibrium. Hence, we have the following result.

**Proposition 2.** *In an all-pay auction with private values and resale, if the resale is conducted through an optimal auction, the symmetric equilibrium strategy for the first-stage players is given by  $\beta_o$ .*

The expected revenue to the seller, denoted  $R_o$ , is

$$R_o = N \int_0^{\omega} \beta_o(x) f(x) dx = \int_0^{\omega} N(N-1)(1-F(t))F(t)^{N-2} f(t) \Pi_0^*(r^*(t)) dt \quad (18)$$

Unfortunately, it is not analytically possible to compare  $R_o$  with  $R$  without assuming a specific distribution function. However, we do know that, when the winner uses an optimal auction to resell the prize, the revenue equivalence result does not remain anymore. Without market separation, the final allocation is efficient in the symmetric equilibrium. With resale and market separation, the final allocation may not be efficient since there is positive probability that the prize ends up with unsold in the second stage. The winner trades off allocative efficiency in order to extract as much surplus as possible. Hence, the allocation rule is violated and the revenue equivalence is broken. However, the expected surplus extracted by

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<sup>7</sup>Of course, in equilibrium, no one will get deceit and everyone bids truthfully.

the winner through an optimal auction is no less than that extracted through a second-price auction. Since the total surplus is fixed, we have

**Corollary 1.** *When the first-stage winner chooses different resale mechanisms, the expected revenue for the initial seller satisfies*

$$R_o \leq R$$

Generally, given these two resale mechanisms, the initial seller never gains through the existence of resale opportunities. If she has access to the whole market, she should include all potential  $N + M$  players in the first stage to secure as much surplus as possible. If the resale is conducted through a second-price auction, the initial seller does not lose anything in terms of expected revenue; if the resale is conducted through an optimal auction, she never gains and may lose in terms of expected revenue.

### 3 An Detailed Example

Here we consider a simple example to show our main results. Suppose, each first-stage bidder has independent private value drawn from the identical uniform distribution over  $[0, 1]$  and each aftermarket buyers has independent private value drawn from the identical uniform distribution over  $[0, 1]$ . We maintain all other assumptions as before.

#### 3.1 Second-Price Resale Mechanism

##### 3.1.1 Equilibrium

Let's solve the game through backward induction. We start from the second stage. Suppose there exists a monotonic increasing equilibrium bidding strategy for the first-stage bidders— $\beta$ . Assume that the winning bid is  $\bar{b}$ , and aftermarket buyers infer that the winning bidder's private valuation  $\bar{v} = \beta^{-1}(\bar{b})$ . Hence,  $\bar{v}$  will serve as the reserve price in the second-price

auction. All buyers valuing the prize below  $\bar{v}$  will not submit positive bids, and all buyers valuing the prize above  $\bar{v}$  will bid their valuations. The expected payment of a bidder with value  $v \geq \bar{v}$  is given by

$$P = \bar{v}G(\bar{v}) + \int_{\bar{v}}^v yg(y)dy \quad (19)$$

where  $G$  is the distribution of  $y = \max_{m \neq j} v_m$ , with a density  $g$ . In this example,  $G(\bar{v}) = \bar{v}^{M-1}$ .

Therefore,

$$P = \bar{v}^M + \int_{\bar{v}}^v y(M-1)y^{M-2}dy = \frac{1}{M}\bar{v}^M + \frac{M-1}{M}v^M \quad (20)$$

The ex ante expected payment of a bidder  $j$  is

$$E[P] = \int_{\bar{v}}^1 Pf(v)dv = \frac{1}{M}\bar{v}^M - \frac{2}{M+1}\bar{v}^{M+1} + \frac{M-1}{M(M+1)} \quad (21)$$

Suppose the winner's private value is  $v$  and bids as if her true value is  $\bar{v}$ , then the overall expected payoff of her from resale is:

$$\Pi_0 = M \times \left[ \frac{1}{M}\bar{v}^M - \frac{2}{M+1}\bar{v}^{M+1} + \frac{M-1}{M(M+1)} \right] + F(\bar{v})^M v \quad (22)$$

Therefore, in the first stage, suppose all other bidders follow  $\beta$ , a bidder learns his private value  $v$ , and submits  $b$  to maximize

$$H(\beta^{-1}(b))\Pi_0 - b \quad (23)$$

where  $H$  is the distribution of the highest private value among a first-stage bidder's competitors. Maximizing (23) over  $b$  yields:

$$\begin{aligned} & \frac{1}{\beta'(\beta^{-1}(b))} \left[ (M+N-1)\beta^{-1}(b)^{M+N-2} - \frac{2M(M+N)}{M+1}\beta^{-1}(b)^{M+N-1} \right. \\ & \left. + \frac{(M-1)(N-1)}{M+1}\beta^{-1}(b)^{N-2} + (M+N-1)v\beta^{-1}(b)^{M+N-2} \right] = 1 \end{aligned}$$

In equilibrium, we have  $\beta(v) = b$ , hence

$$\begin{aligned} \beta'(v) = & (M+N-1)v^{M+N-1} + (M+N-1)v^{M+N-2} \\ & + \frac{(M-1)(N-1)}{M+1}v^{N-2} - \frac{2M(M+N)}{M+1}v^{M+N-1} \end{aligned}$$

Therefore, the symmetric equilibrium is given by

$$\beta(v) = v^{M+N-1} + \frac{M-1}{M+1}v^{N-1} + \frac{N-1-M^2-MN}{(M+1)(M+N)}v^{M+N} \quad (24)$$

It is trivial to show that  $\beta'(v) > 0$  for any  $v \in (0, 1]$ ,  $\beta(0) = 0$  and  $\beta(1) = \frac{M+N-1}{M+N} < 1$ . Therefore,  $\beta(\cdot)$  is strictly increasing, differentiable, and continuous. Moreover, it is easy to verify that  $\beta$  is indeed a symmetric equilibrium in the all-pay auction with resale.

It is easy to show that, if there is no secondary market, the symmetric equilibrium strategy is given by<sup>8</sup>

$$\tilde{\beta}(v) = \frac{N-1}{N}v^N \quad (25)$$

and if there is no market separation, i.e., the initial seller could gather all potential  $M+N$  bidders into an all-pay auction, the symmetric equilibrium strategy is given by

$$\hat{\beta}(v) = \frac{M+N-1}{M+N}v^{M+N} \quad (26)$$

Note if there is no resale market, i.e.,  $M = 0$ ,

$$\beta(v) = \tilde{\beta}(v) = \hat{\beta}(v) = \frac{N-1}{N}v^N \quad (27)$$

If the first-stage players anticipate that there will be no interested aftermarket buyers, they will stop signalling their private information and compete as they would in an all-pay auction

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<sup>8</sup>The easiest way to derive the equilibrium bidding strategy is to apply revenue equivalence theorem.

without resale.

### 3.1.2 Revenues

The interesting questions we would like to answer are: when it is optimal to allow resale opportunities, when it is optimal to forbid resale, and when it is optimal to organize a whole auction including all potential bidders?

Let  $R_N$  denote the expected revenue for the seller from an all-pay auction with resale,  $\tilde{R}_N$  denote the expected revenue for the seller from an all-pay auction without resale,  $\hat{R}_{M+N}$  denote the expected revenue for the seller to organize an all-pay auction with all potential bidders (without resale). Then we have

$$R_N = N \int_0^1 \beta(v) dv = \frac{M+N-1}{M+N+1} \quad (28)$$

$$\tilde{R}_N = N \int_0^1 \tilde{\beta}(v) dv = \frac{N-1}{N+1} \quad (29)$$

$$\hat{R}_{M+N} = (M+N) \int_0^1 \hat{\beta}(v) dv = \frac{M+N-1}{M+N+1} \quad (30)$$

Clearly  $R_N > \tilde{R}_N$ . In an all-pay auction with resale, the first-stage bidders not only compete for the prize, but also compete for the reselling opportunity in the secondary stage. The potential benefit from resale induces those first-stage bidders to bid more aggressively, thus increase the seller's expected revenue.

Second,  $\hat{R}_{M+N} > \tilde{R}_N$ . The intuition is straightforward. The presence of additional bidders make the competition fiercer and thus bidders tend to bid more aggressively. Moreover, the payment rule also implies that the seller could collect more revenue from additional bidders.

Third,  $R_N = \hat{R}_{M+N}$ . This could be explained by the celebrated revenue equivalence theo-

rem in auction theory.<sup>9</sup> In our independent private value framework, even if there is a resale market, the allocation is the same and the prize always goes to the bidder with highest valuation. The bidder with lowest possible valuation always bid zero and thus his expected payoff is zero. Therefore, the revenue equivalence theorem implies that the expected revenue for the seller is the same, regardless of the separation of two markets and resale opportunities.

Now let us consider bidders' equilibrium behaviors in the above three auctions. By simple manipulation, we have

$$\beta(v) - \tilde{\beta}(v) = v^{M+N} \left[ \frac{1}{v} - \frac{M^2 + MN + 1 - N}{(M+1)(M+N)} \right] + v^N \left[ \frac{M-1}{v(M+1)} - \frac{N-1}{N} \right] \quad (31)$$

Therefore, as long as  $\frac{M-1}{v(M+1)} > \frac{N-1}{N}$ , we have  $\beta(v) > \tilde{\beta}(v)$ . It is not clear whether we have  $\beta(v) > \tilde{\beta}(v)$  over  $[0, 1]$ , though we do have  $\int_0^1 \beta(v) dv > \int_0^1 \tilde{\beta}(v) dv$ . However, we do know that both bidding strategies are strictly increasing and continuous, then if we can show  $\beta(1) \geq \tilde{\beta}(1)$ , then we have  $\beta(v) > \tilde{\beta}(v)$  over  $[0, 1]$ . Clearly,  $\tilde{\beta}(1) = \frac{N-1}{N}$ ; and  $\beta(1) = \frac{M+N-1}{M+N} > \frac{N-1}{N}$ . Therefore, allowing resale opportunities will make the first-stage bidders compete more aggressively.

Moreover, we have

$$\beta(v) - \hat{\beta}(v) = v^{M+N} \left[ \frac{1}{v} + \frac{M-1}{v^{M+1}(M+1)} - \frac{2M}{M+1} \right] \geq 0 \quad (32)$$

The last inequality follows since

$$\frac{1}{v} + \frac{M-1}{v^{M+1}(M+1)} - \frac{2M}{M+1} \geq 1 + \frac{M-1}{M+1} - \frac{2M}{M+1} = 0$$

Therefore, each bidder bids more aggressively if the initial seller strategically excludes some

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<sup>9</sup>Riley and Samuelson (1981) and Myerson (1981) prove a revenue-equivalence theorem which states that, under the assumptions of risk neutrality, independence of valuations, and symmetry among bidders, all auctions rules within a broad class generate the same expected revenue for the seller if, in equilibrium, the incentive to participate in the auction remain the same.



bidders from the first-stage competition and forces them to compete in the second stage. This confirms with intuition underlying the revenue equivalence result. In an all-pay auction with resale, there are less bidders in the first-stage competition but the seller still gets the same expected revenue as organizing an auction including all potential bidders. Therefore, for the revenue equivalence to hold, it must be true that the first-stage bidders bid more aggressively with resale opportunities, since any extra surplus from the second-stage goes to the winner. In equilibrium, the seller does not lose anything by excluding some bidders from the first-stage competition.

Finally, we have

$$\tilde{\beta}(v) - \hat{\beta}(v) = v^N \left[ \frac{N-1}{N} - \frac{v^M (M+N-1)}{M+N} \right] \quad (33)$$

Therefore,  $\tilde{\beta}(v) > \hat{\beta}(v)$  if and only if  $v < \sqrt[M]{\frac{(N-1)(M+1)}{N(M+N-1)}}$ . Without resale opportunities, a bidder with relatively low value will bid less aggressively if more competitors are included in all-pay auctions. With more competitors, a bidder will bid more aggressively only if he has a relatively high value. This finding is related to certain experimental findings about bidders' behaviors in all-pay auctions. Noussair and Silver (2006) observe extensive use of a dichotomous bidding strategy: for relatively low valuations, bids of zero are common, while for high valuations, bids exceeding equilibrium levels are typical.<sup>10</sup>

Recall that, with independent private values identically drawn from uniform distribution over  $[0, 1]$ , a bidder in first-price auction will bid  $\frac{N-1}{N}v$  if there are  $N$  bidders and will bid more aggressively with increasing  $N$ .<sup>11</sup> Clearly,  $\frac{N-1}{N}v \geq \frac{N-1}{N}v^N$  for all  $v \in [0, 1]$ . This is because of the unconditional payment rule in the all-pay auctions. Regardless winning or

<sup>10</sup>The unconditional payment rule has a bite here. In standard winner-pay auctions, we do not observe such dichotomous bidding strategies.

<sup>11</sup>However, in all-pay auctions, not everyone will bid more aggressively with more competitors. Only those with relatively high valuations will compete more aggressively, but those with relatively low valuations will compete less aggressively.

losing, each bidder has to pay his bid in all-pay auctions. Contrasting to the conditional payment rule in first-price auctions where only the winner is required to pay, bidders in all-pay auctions further shade their bids in order to insure against any potential loss in case of losing the auction. This could also be explained by revenue equivalence result. The expected revenues are the same across both auction forms, then bidders in all-pay auctions must bid less aggressively than those in first-price auctions since everyone needs to pay their bids in all-pay auctions.

### 3.1.3 Optimal Exclusion

Since the initial seller benefits from existence of an active resale market, it may be natural to ask whether she has incentive to exclude some buyers from the primary competition to create a resale market. From previous analysis, if the first-stage winner use a second-price auction to resell the prize, the expected revenue to the initial seller is the same if she excludes some bidders from the first-stage competition and force them into a resale market. In some cases, the first-stage winner may have access to information or commitment technologies enabling him to extract more surplus from other buyers than the initial seller can. In such cases the seller may benefit from letting buyers compete for the right to sell in the second stage, thereby passing through at least some of the added surplus extraction realized in the second stage. We address this problem through the following example.<sup>12</sup>

**Example 1.** *Suppose there are  $N$  risk-neutral bidders, each having independent private valuation from the identical uniform distribution  $F$  on  $[0, 1]$ .*

For simplicity, there are no new entrant arriving exogenously, but the initial seller may exclude bidders from the primary competition, forcing them into the resale market. First, by applying the revenue equivalence theorem, the expected revenue ( $R_N$ ) is the expected value

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<sup>12</sup>See Haile (1999) for a similar analysis of second-price auctions.

of the second highest valuation among  $N$  bidders. Hence, we have

$$R_N = E[Y_1] = \frac{N-1}{N+1} \quad (34)$$

Now suppose one of the bidders is excluded at random, this creates an opportunity for the winner to resell in the secondary market. Suppose the winner of the primary competition could extract a share  $\alpha$  of any surplus in the resale market. Then, upon winning, the endogenously-determined valuation becomes:

$$\hat{v}(x) = xF(x) + \int_x^1 [x + \alpha(z-x)]f(z)dz = (1-\alpha)x + \frac{\alpha}{2}(1+x^2) \quad (35)$$

Solving the two-stage game as before, we get the symmetric equilibrium strategy

$$\beta_{N-1} = \int_0^x \hat{v}(t)t^{N-2}dt \quad (36)$$

Therefore,

$$R_{N-1} = (N-1) \int_0^1 \int_0^x \hat{v}(t)t^{N-2}dtdx = \frac{N^2 - N - 2 + 3\alpha}{N(N+1)} \quad (37)$$

Then  $R_{N-1} - R_N = \frac{3\alpha-2}{N(N+1)}$ . Therefore, as long as  $\alpha > \frac{2}{3}$ , i.e., the first-stage winner could extract at least two thirds of resale surplus, the initial seller could benefit from excluding one bidder from the first-stage competition and creating an active resale market.<sup>13</sup>

Assuming complete information, Baye et al. (1993) present an interesting *exclusion principle*: a revenue maximizing politician may find it in her best interest to exclude lobbyists with valuations above a threshold from participating in the all-pay auction. The intuition is quite straightforward. The presence of high-value competitors makes the playing field unequal and deter low-value competitors from bidding aggressively. The exclusion makes the

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<sup>13</sup>It is not so clear whether the initial seller benefits from excluding more than one bidder from the first-stage competition when the first-stage winner could extract at least two thirds of resale surplus.

competition more even and bidders submit higher bids, which in turn increases the initial seller's expected revenue.

This optimal exclusion principle applies not only for all-pay auctions. Haile (1999) derives similar result for a second-price auction followed by resale. In particular, he shows that as long as the first-stage winner extracts no less than two thirds of resale surplus, the initial seller could benefit from excluding one bidder from the first-stage competition. Bose and Deltas (1999) study an auction with two distinct type of potential bidders: consumers who bid for their own consumption and speculators who bid for resale. They show that, if the speculators have access to a larger market of consumers than the auctioneer, then the auctioneer may prefer to prevent the consumers from participating in the auction.

### 3.2 Optimal Resale Mechanism

Now suppose resale takes place via an optimal auction characterized by Myerson (1981). That is, the first-stage winner organizes an optimal auction on the secondary market. Different from Myerson's setting, the seller of the secondary market, i.e., the first-stage winner, values the object at a positive amount instead of zero. Again we solve this two-stage game through backward induction.

Consider the second stage. Suppose there exists a monotonic increasing equilibrium bidding strategy for the first-stage bidders- $\beta_o$ . From Myerson (1981), we know that a second-price auction with a properly chosen reserve price will be an optimal auction. We focus on the regular case in which every aftermarket buyer's virtual valuation  $\psi(\cdot)$  is an increasing function of the true value  $v_i$ . Define

$$\psi(v_i) \equiv v_i - \frac{1 - F(v_i)}{f(v_i)} \quad (38)$$

to be the virtual valuation of a buyer with value  $v_i$ .<sup>14</sup> In our special case with symmetric uniform distribution,  $\psi(v_i) = 2v_i - 1$ . Hence, if a first-stage bidder with value  $v$  wins the competition, he will set the reserve price equal to  $\psi^{-1}(v) = \frac{1+v}{2}$ . Let  $r \equiv \psi^{-1}(v)$ . In the second stage, the expected payment of a bidder with value  $v \geq r$  is given by

$$P = rG(r) + \int_r^v yg(y)dy \quad (39)$$

where  $G$  is the distribution of  $y = \max_{m \neq j} v_m$ , with a density  $g$ . In this example,  $G(r) = r^{M-1}$ .

Therefore,

$$P = r^M + \int_r^v y(M-1)y^{M-2}dy = \frac{1}{M}r^M + \frac{M-1}{M}v^M \quad (40)$$

The ex ante expected payment of a bidder  $j$  is

$$\begin{aligned} E[P] &= \int_r^1 Pf(v)dv \\ &= \int_r^1 \left[ \frac{1}{M}r^M + \frac{M-1}{M}v^M \right] dv \\ &= \frac{1}{M}r^M - \frac{2}{M+1}r^{M+1} + \frac{M-1}{M(M+1)} \end{aligned}$$

Suppose the winner's private value is  $x$  and sets reserve price equal to  $r$  in an optimal auction, then the overall expected payoff of her from resale is:

$$\Pi_0 = M \times \left[ \frac{1}{M}r^M - \frac{2}{M+1}r^{M+1} + \frac{M-1}{M(M+1)} \right] + F(r)^M x \quad (41)$$

Therefore, in the first stage, suppose all other bidders follow  $\beta_o$ , a bidder learns his private

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<sup>14</sup>The virtual valuation can be interpreted as a marginal revenue, if we consider the seller as a monopolist using third-degree price discrimination. Bulow and Roberts (1989) show that the main results and elegant techniques developed in optimal auction design can be reinterpreted in the language of standard micro theory.

value  $x$ , and submits  $b$  to maximize

$$H(\beta_o^{-1}(b))\Pi_0 - b \quad (42)$$

where  $H$  is the distribution of the highest private value among a first-stage bidder's competitors. Maximizing (42) over  $b$ , the first-order condition yields:

$$\beta'_o(x) = \frac{(N-1)(M-1)}{M+1}x^{N-2} + \frac{2(N-1)}{M+1}x^{N-2}\left(\frac{x+1}{2}\right)^{M+1} \quad (43)$$

Since  $\beta_o(0) = 0$ , we have

$$\beta_o(x) = \frac{M-1}{M+1}x^{N-1} + \frac{(N-1)}{(M+1)2^M} \int_0^x t^{N-2}(t+1)^{M+1} dt \quad (44)$$

As previous analysis, we derive the expected revenue for the initial seller when the first-stage winner uses an optimal auction to resell the prize to aftermarket buyers. Let  $R_o$  denote the expected revenue to the initial seller in this case, we have

$$\begin{aligned} R_o &= N \int_0^1 \beta_o(x) dx \\ &= \frac{M-1}{M+1} + \frac{N(N-1)}{(M+1)2^M} \int_0^1 \left( \int_0^x t^{N-2}(t+1)^{M+1} dt \right) dx \\ &= \frac{M-1}{M+1} + \frac{N(N-1)}{(M+1)2^M} \int_0^1 \left( \int_t^1 dx \right) t^{N-2}(t+1)^{M+1} dt \\ &= \frac{M-1}{M+1} + \frac{N(N-1)}{(M+1)2^M} \int_0^1 t^{N-2}(1-t)(t+1)^{M+1} dt \end{aligned}$$

where the third equality follows as we interchange the order of integration.

However, given the above expression of  $R_o$ , it is too complicated to compare it with  $R_N$ , i.e., the expected revenue for the initial seller when the first-stage winner uses a second-price auction to resell the prize. Roughly speaking, the revenue equivalent result may not hold any

longer. We use the following example to get some rough ideas.

**Example 2.** *There are two first-stage players ( $N = 2$ ) and two aftermarket buyers ( $M = 2$ ). All players have independent private value drawn from a uniform distribution over  $[0, 1]$ .*

According to those equilibrium strategies we characterize above,  $R_N = R_{M+N} = \frac{M+N-1}{M+N+1} = \frac{3}{5}$ . Now let's calculate  $R_o$ . Plugging  $N = M = 2$  into the expression of  $R_o$ , we have

$$R_o = \frac{1}{3} + \frac{2}{12} \int_0^1 (1-t)(t+1)^3 dt = \frac{11}{20} < \frac{3}{5} = R_N \quad (45)$$

Before we draw the conclusions, let's consider another example.

**Example 3.** *There are three first-stage players ( $N = 3$ ) and two aftermarket buyers ( $M = 2$ ). All players have independent private value drawn from a uniform distribution over  $[0, 1]$ .*

Repeating the same procedure, we have  $R_N = \frac{2}{3}$ . As for  $R_o$  in this case, we have

$$R_o = \frac{1}{3} + \frac{1}{2} \int_0^1 t(1-t)(t+1)^3 dt = \frac{19}{30} < \frac{2}{3} = R_N \quad (46)$$

**Example 4.** *There are two first-stage players ( $N = 2$ ) and three aftermarket buyers ( $M = 3$ ). All players have independent private value drawn from a uniform distribution over  $[0, 1]$ .*

Repeating the same procedure, we have  $R_N = \frac{2}{3}$ . As for  $R_o$  in this case, we have

$$R_o = \frac{1}{2} + \frac{1}{16} \int_0^1 (1-t)(t+1)^4 dt = \frac{297}{480} < \frac{2}{3} = R_N \quad (47)$$

**Remark 3.** *From our simple examples, we see that the expected revenue collected by the initial seller declines when the first-stage winner resells the prize through an optimal auction (in Myerson's sense) instead of a second-price auction. Moreover, the revenue equivalent result does not hold any more. What is the intuition underlying this result? Using an optimal*

*auction, the first-stage winner trades expected surplus with allocative efficiency. The ex post allocation is efficient when the first-stage winner resells the prize through a plain second-price auction, since it is always the player with the highest valuation who gets the prize. However, with optimal resale, though the first-stage winner could collect more expected revenue, the ex post allocation may not be efficient, since the prize may end up with unsold even if it is efficient to trade. Since the total surplus is fixed, the expected surplus extracted by the winner through an optimal auction is no less than that extracted through a second-price auction, the expected revenue for the initial seller is the opposite.*

Given the total number of players, when the first-stage winner resells the prize through an optimal auction, there may be certain difference regarding how many players the initial seller excludes from the primary competition. From our simple examples, we see that the initial seller may benefit from excluding less players from the primary competition. Continuing this argument, the initial seller should not exclude any player from the primary competition, since that would decrease her expected revenue. It seems that this conclusion contradicts with the result we present in Example 1. Actually it is not the case. In Example 1, though the first-stage winner could extract more surplus from aftermarket buyers, the ex post allocation is still efficient.

To summarize, when the first-stage winner uses a plain second-price auction to resell the prize, the initial seller gets the same expected revenue; when the first-stage winner uses an optimal auction to resell the prize, the initial seller gets less expected revenue. Moreover, when the first-stage winner has ability to extract at least two thirds of resale surplus, the initial seller benefits from excluding one player from the primary competition.



## 4 Conclusions

We study all-pay auctions with resale opportunities when players have independent private values. We enrich a static all-pay auction model by introducing resale opportunities, wishing to investigate how resale changes primary players' strategic behaviors. Given existence of a resale market, the primary players will compete more aggressively over an indivisible prize. Not only do they compete over the prize itself, but also they compete over the selling right in a resale market since there may be new market entrants. The existence of a resale market establishes a link between players' first-stage bidding and his or her resale surplus, leaving first-stage players' valuation endogenously determined.

In the resale market, we formalize two resale mechanisms: a second-price auction and an optimal auction. Certainly different resale mechanism has different impact on primary players' strategic behaviors. In both cases, we characterize the symmetric equilibria for all-pay auctions with private values and resale. Based on those equilibria, we derive a revenue-ranking result for all-pay auctions with and without resale opportunities in each resale mechanism. Using an illustrative example, we also compare initial seller's expected revenues when the first-stage winner uses different resale mechanisms. We derive an interesting revenue-equivalence result when the first-stage winner conducts resale through a second-price auction. In particular, the initial seller does not lose anything in terms of expected revenue due to the market separation. However, this result does not remain if the first-stage winner conducts resale through an optimal auction. The tradeoff between expected revenue and allocative efficiency reduces the initial seller's expected revenue. Hence, if there is no market separation, the initial seller does not benefit from excluding some players from primary competition and forcing them into a resale market.

However, depending on the first-stage winner's ability of extracting surplus in the resale stage, the initial seller may benefit from excluding some players from the primary competition

thus creating a resale market. Extremely, if the first-stage winner has full bargaining power and thus extract all resale surplus, they will compete even more aggressively and leave more surplus to the initial seller. Of course this is a very strong assumption, meaning the secondary players do not have any information rent. How this works needs more investigation and more real-life examples are needed to support such an argument.

One future research concerns the endogenous entry of players if we allow the first-stage losing players to compete in the resale market. Then a player may strategically bid to lose the first-stage auction and buy the prize in the resale market. Using a two-player model with public values, Pagnozzi (2007) shows why a strong bidder may prefer to drop out of the auction before the price has reached her valuation and acquire the prize in the aftermarket. The intuition is that a strong bidder may be in a better bargaining position in the aftermarket if her rival won at a relatively low price. Extending the analysis to more than two players with private values will be an interesting research direction.

Moreover, the main results in this paper would serve as natural theoretical prediction for experimental study of all-pay auctions with resale. Noussair and Silver (2006) investigate all-pay auctions with independent private values without resale. A natural extension would be to study the same model but with resale opportunities.

## Appendix A

*Proof of the Revenue Ranking Result:  $R = \hat{R} \geq \tilde{R}$*

*Proof.* To show that  $R = \hat{R}$ , let us first calculate  $E[\beta(x)]$ .

$$\begin{aligned}
E[\beta(x)] &= \int_0^\omega \beta(x)f(x)dx \\
&= \int_0^\omega MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y)dyf(x)dx + \int_0^\omega MxF(x)^{M+N-2}f(x)dx \\
&\quad - \int_0^\omega (M-1)xF(x)^{M+N-1}f(x)dx - \int_0^\omega \int_0^x F(t)^{M+N-1}dtf(x)dx \\
&= \int_0^\omega \int_x^\omega \frac{M}{N}y(1-F(y))g(y)dydF(x)^N + \int_0^\omega MxF(x)^{M+N-2}f(x)dx \\
&\quad - \int_0^\omega (1-F(x))F(x)^{M+N-1}dx - (M-1) \int_0^\omega xF(x)^{M+N-1}f(x)dx \\
&= \int_0^\omega \frac{M(M-1)}{N}F(x)^{M+N-2}x(1-F(x))f(x)dx + \int_0^\omega xd[(1-F(x))F(x)^{M+N-1}] \\
&\quad + \int_0^\omega MxF(x)^{M+N-2}f(x)dx - \int_0^\omega (M-1)xF(x)^{M+N-1}f(x)dx \\
&= \int_0^\omega \frac{M(M+N-1)}{N}F(x)^{M+N-2}f(x)xdx - \int_0^\omega \frac{(M+N)(M-1)}{N}F(x)^{M+N-1}f(x)xdx \\
&\quad - \int_0^\omega [(M+N-1)F(x)^{M+N-2}(1-F(x))f(x) - F(x)^{M+N-1}f(x)]xdx \\
&= \int_0^\omega \frac{M(M+N-1)}{N}F(x)^{M+N-2}f(x)xdx - \int_0^\omega \frac{(M+N)(M-1)}{N}F(x)^{M+N-1}f(x)xdx \\
&\quad + \int_0^\omega (M+N-1)F(x)^{M+N-2}f(x)xdx - \int_0^\omega (M+N)F(x)^{M+N-1}f(x)xdx \\
&= \int_0^\omega \frac{(M+N)(M+N-1)}{N}F(x)^{M+N-2}f(x)xdx \\
&\quad - \int_0^\omega \frac{(M+N)(M+N-1)}{N}F(x)^{M+N-1}f(x)xdx \\
&= \int_0^\omega \frac{(M+N)(M+N-1)}{N}(1-F(x))F(x)^{M+N-2}f(x)xdx
\end{aligned}$$

where the third and fourth equality result by repeating the procedure of integrating by parts, and combining similar items.

Therefore, we have

$$R = NE[\beta(x)] = \int_0^{\omega} (M+N)(M+N-1)(1-F(x))F(x)^{M+N-2}f(x)xdx$$

The above expression is nothing but the expectation of second-highest order statistics among  $(M+N)$  values. In particular, let  $Y_2^{(M+N)}$  denote the second-highest order statistics among  $(M+N)$  values, where each value is an independent draw from identical distribution  $F(\cdot)$  with density  $f$ . Then we have

$$\hat{G}(y) = F(y)^{M+N} + (M+N)F(y)^{M+N-1}(1-F(y)) = (M+N)F(y)^{M+N-1} - (M+N-1)F(y)^{M+N}$$

The associated probability density function is

$$\hat{g}(y) = (M+N)(M+N-1)(1-F(y))F(y)^{M+N-2}f(y)$$

Therefore,

$$R = \int_0^{\omega} (M+N)(M+N-1)(1-F(x))F(x)^{M+N-2}f(x)xdx = E[Y_2^{(M+N)}]$$

Recall that

$$\begin{aligned} \hat{R} &= (M+N) \int_0^{\omega} \hat{\beta}(x)f(x)dx \\ &= (M+N) \int_0^{\omega} x(1-F(x))\hat{g}(x)dx \\ &= \int_0^{\omega} (M+N)(M+N-1)(1-F(x))F(x)^{M+N-2}f(x)xdx \\ &= E[Y_2^{(M+N)}] \end{aligned}$$

Obviously, we obtain  $R = \hat{R} = E[Y_2^{(M+N)}]$ . Hence, the revenue equivalence remains.

Recall that

$$\begin{aligned}
\tilde{R} &= N \int_0^\omega \tilde{\beta}(x) f(x) dx \\
&= N \int_0^\omega x(1-F(x)) \tilde{g}(x) dx \\
&= \int_0^\omega N(N-1)(1-F(x)) F(x)^{N-2} f(x) x dx \\
&= E[Y_2^{(N)}]
\end{aligned}$$

where  $Y_2^{(N)}$  is the second-highest order statistics of  $N$  values.

Clearly we have  $\hat{R} \geq \tilde{R}$  since

$$\hat{R} = E[Y_2^{(M+N)}] \geq E[Y_2^{(N)}] = \tilde{R}$$

Therefore, we obtain

$$R = \hat{R} \geq \tilde{R}$$

□

## Appendix B

*Simplify the bidding strategy for second-price resale mechanism*

Recall that

$$\begin{aligned}
\beta(x) &= \int_0^x MF(t)^{M+N-2} dt - \int_0^x MF(t)^{M+N-1} dt \\
&+ \int_0^x M(N-1)tF(t)^{M+N-3} f(t) dt - \int_0^x (N-1)(M-1)tF(t)^{M+N-2} f(t) dt \\
&+ \int_0^x M(N-1)F(t)^{N-2} f(t) \int_t^\omega y(1-F(y))g(y) dy dt
\end{aligned}$$

To simplify the bidding strategy, we show

$$\begin{aligned}
& \int_0^x M(N-1)F(t)^{N-2}f(t) \int_t^\omega y(1-F(y))g(y)dydt \\
&= \int_0^x M \int_t^\omega y(1-F(y))g(y)dydF(t)^{N-1} \\
&= M[\int_t^\omega y(1-F(y))g(y)dyF(x)^{N-1}]|_0^x - \int_0^x MF(t)^{N-1}d[\int_t^\omega y(1-F(y))g(y)dy] \\
&= MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y)dy + \int_0^x tF(t)^{N-1}(1-F(t))g(t)dt
\end{aligned}$$

Hence,

$$\begin{aligned}
\beta(x) &= MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y)dy + \int_0^x tF(t)^{N-1}(1-F(t))g(t)dt \\
&+ \int_0^x MF(t)^{M+N-2}dt - \int_0^x MF(t)^{M+N-1}dt \\
&+ \int_0^x M(N-1)tF(t)^{M+N-3}f(t)dt - \int_0^x (N-1)(M-1)tF(t)^{M+N-2}f(t)dt \\
&= MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y)dy \\
&+ \int_0^x MF(t)^{M+N-2}dt + \int_0^x M(M+N-2)tF(t)^{M+N-3}f(t)dt \\
&- \int_0^x MF(t)^{M+N-1}dt - \int_0^x (M-1)(M+N-1)tF(t)^{M+N-2}f(t)dt \\
&= MF(x)^{N-1} \int_x^\omega y(1-F(y))g(y)dy - \int_0^x F(t)^{M+N-1}dt \\
&+ Mx F(x)^{M+N-2} - (M-1)x F(x)^{M+N-1}
\end{aligned}$$

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