Inflation, Entrepreneurship and Growth

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August 29, 2012†

Abstract

There is a long tradition of linking entrepreneurship to economic growth. This paper studies the effects of monetary policy, in the sense of fully anticipated inflation, on entrepreneurship and its impact on economic growth. With the working capital requirement for labor inputs, I find that unless labor supply is perfectly elastic, inflation induces more entrepreneurship. When entrepreneurship is the driving force of long run economic growth, inflation can increase growth, while its impact on output at given productivity level. This channel is novel and has many testable implications. I also discuss how this model can provides a theoretical rationale for the empirical findings that growth can increase with long-run inflation at low levels of inflation.

Key words: Inflation; Entrepreneurship; Growth; Cash Holdings of Firms; Occupational Choice

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1 Introduction

The idea that economic growth is driven by entrepreneurial innovations can be traced back to Schumpeter (1934). There is an extensive literature in economics linking economic growth to entrepreneurship. These works tend to share a common implication: policies or institutional changes that encourage entrepreneurial activities lead to higher growth. For example, while King and Levine (1993) have argued that the higher the quality of financial intermediaries the more potential entrepreneurs with technology advancing projects will be financed, generating faster economic growth, Aghion, et al. (2009) emphasize entrepreneurs' ability to adopt frontier technologies in promoting economic growth. In a similar spirit, this paper studies the effect of another policy, in the sense of fully anticipated long run inflation, on entrepreneurship and its impact on economic growth.

The baseline model of this paper deviates from the standard neoclassical model in the following two ways: first, I assume that production cost needs to be paid before entrepreneurs receive their revenue, or to say, I put a working capital requirement for them to buy labor inputs; second, agents in the model can choose between two types of occupations (entrepreneurs and workers). In this baseline model, I show that when labor supply is inelastic, inflation induce more entrepreneurship, and vice versa. The basic intuition of these results is that we can think of inflation as a tax on monetary transactions, so when supply is inelastic, the suppliers bear the cost of inflation tax, and when supply is elastic, the buyers bear the cost of inflation. These results hold for with or without borrowing constraint.

Then I extend this baseline model to incorporate endogenous growth in a way similar to Chiu, Meh and Wright (2011). I assume each entrepreneur has some probability of coming up some innovation which boost their profit temporarily but will get absorbed into the aggregate productivity in the long run. Therefore more entrepreneurs means higher long-term growth. With this extension the implication is straight forward.

The next step that I will do is to incorporate heterogeneous entrepreneurial ability to innovate. It is natural to think in equilibrium the entrepreneurs have higher ability than the agents who choose not to become entrepreneurs. I will focus on the case where the individual labor supply elasticity is low: at low levels of inflation, the marginal entrepreneur is of high ability thus changes of long-run inflation will have big effect on economic growth while at high levels of inflation the marginal entrepreneur is of low ability thus the changes of long-run inflation only has small effect on growth. Combined with other channels of growth, this can potentially generate patterns such that inflation increases growth at low level but has negative effect at high level of inflation.

This paper is much related to monetary economics. There is no question that long-run inflation is one of the most important and robust factors affecting long-run economic growth[1]. Economists have proposed various mechanisms. The question is how does it affect growth. Tobin (1965) argued that inflation can enhance accumulation of capital instead of holding money[2]. Subsequent studies, such as Chari, Jones and Manuelli (1995), examined endogenous growth models how inflation affects growth by distorting the accumulation of either (both) physical or (and) human capital. See Gillman and Kejak (2005) for an extensive survey. Even today, there are still substantial interests in addressing this question. Berentsen, Breu and Shi (2009) developed a model where inflation negatively affects growth by discouraging decentralized trades of innovation goods which are used to increase labor productivity. Chiu, Meh and Wright (2011) studied the decentralized trade between innovators, who come up with new ideas, and entrepreneurs, who are better in implementing these ideas. The endogenous growth models generally predict that inflation negatively affects growth at all inflation levels[3].

Compared with the existing endogenous growth theories that study the effect of long-run inflation, the mechanism in this paper is quite different. In general, these theories assume some trades that are directly good for growth involves money[4]. Inflation discourages such trades thus negatively affects growth. Here, the trade that requires money (labor in-

[1]For example, Durlauf, Kourtellos and Tan (2009) found long-run inflation to be one of the robust predictors of economic growth.

[2]The literature on money and capital is large, including Sidrauski (1967a, 1967b), Stockman (1981), Ireland (1994), Aruoba, Waller and Wright (2008) and many more.

[3]In Chari, Jones and Manuelli (1995), the correlation of long-run inflation and growth can be positive only at extremely high inflation levels.

[4]For example, inflation lowers consumption of cash good thus lowers the return of human capital as in Chari, et al. (1995); or inflation discourages trade of innovation goods as in Berentsen, et al. (2009).

puts) is not directly related to growth. Inflation affects growth by influencing occupational choice. Another important difference is that this paper rationalizes the stylized fact that inflation can enhance growth at low inflation rates.

On the empirical side, it is indeed widely accepted that too much long-run inflation is harmful for economic growth. However, it has also been advanced that the relationship between economic growth and inflation might depend on the inflation level. The so-called "threshold effect" has been confirmed by many authors, such as Sarel (1996), Ghosh and Philips (1998), Benhabib and Spiegel (2009) and Gillman and Kejak (2005). They found that the correlation between long-run inflation and economic growth is negative above some threshold and insignificant or even slightly positive below the threshold. Further more, Benhabib and Spiegel (2009) documented a significant positive relationship between inflation and growth in ranges of moderate to negative inflation while Lopez-Villavicencio and Mignon (2011) found similar patterns for advanced countries and a non-significant correlation for emerging ones[5]. Even if we ignore the positive correlation of inflation and growth at low inflation rates and just look at the threshold effect, that is, the correlation is only significantly negative if inflation is higher than a certain cutoff, the evidence still suggests there must be some channel through which inflation can enhance growth and thus counterbalance the existing theoretical channels at low inflation levels.

The rest of the paper is organized as follows. In Section 2, I use the simpliest model setup to illustrate the main mechanism. Section 3 extends the beseline model with endogenous growth and imperfect bank credit. Section 4 concludes with a discussion of potential extensions.

[5]The thresholds for advanced countries and emerging ones are different: 2.7% and 17.5% respectively.

2 Baseline Model with Perfect Bank Credit

2.1 Exogenous Entrepreneurship

There is a unit measure of agents in the economy. Time is discrete and agents live forever. There are n entrepreneurs and $1-n$ workers ($n<1$). There are two kinds of nonstorable consumption goods, a market good, x, and an endowment good, y. The market good is produced by entrepreneurs using labor inputs with the following prodcution function: $Zf(\ell_e)$, where ℓ_e is the labor inputs employed by an entrepreneur. At the end of every period, each agent receives ZY units of the endowment good, where Z is aggregate productivity, which grows exogenously at rate g. The period utility of agents is $log(x)+y-v(\ell_i)$, where v is a convex function, ℓ_w and ℓ_E are the labor supply of workers and entrepreneurs respectively and let $\ell_E=0$[6]. Agents maximize the sum of discounted periodic utility with discount factor $\beta\in(0,1)$.

Final goods x and y are traded at the end of each period competitively but due to limited commitment, entrepreneurs need to pay workers using money at competitive wages before they receive revenue at the end of each period. At the beginning of each period, agents start with money holding m and can borrow or lend in nominal terms in a competitive credit market. Loans will be repayed at the beginning of next period. In this section we assume there is no restriction on how much an agent can borrow from this credit market.

$$\begin{aligned} V^e(m;M,Z) &= \max_{x,y,\ell_e,m'} log(x)+y+\beta V^e(m';M',Z') \\ s.t.\; m' &= ZYp_y+Zpf(\ell_e)+(m-w\ell_e)R+T-xp-yp_y, \end{aligned} \tag{1}$$

where p and p_y are the prices of market goods and endowment good, R is the gross nominal interest rate in the credit market, T is the lump sum transfer received from central bank and M is the aggregate money supply at the beginning of the period. Notice there is

[6]Here utility is quasilinear in y. As in Lagos and Wright (2005), this will make the distribution of money holdings degenerate despite idiosyncratic shocks every period, which will be useful when we discuss innovation shocks received by entrepreneurs later.

no cash in advance constraint for final goods, so if an entrepreneur does not spend all of his money to buy labor inputs he/she will lend the rest of the money in the credit market instead of hold idle money across periods[7]. Therefore $m - w\ell_e$ is the amount of money he lend (borrow if this is negative) in the credit market which will increase his money holding at the beginning of the next period by $(m - w\ell_e)R$. Similarly the value function of workers is

$$\begin{aligned} V^w(m; M, Z) &= \max_{x,y,\ell_W,m'} log(x) + y - v(\ell_w) + \beta V^w(m'; M', Z') \\ s.t.\ m' &= ZYp_y + w\ell_w + mR + T - xp - yp_y, \end{aligned} \tag{2}$$

Similar to Lagos and Wright (2005), assume interior solution for y and plug in y we have the value functions for the types:

$$\begin{aligned} V^e(m; M, Z) &= \max_{x,\ell_e,m'} log(x) + \beta V^e(m'; M', Z') \\ &+ \frac{1}{p_y}[ZYp_y + Zpf(\ell_e) + (m - w\ell_e)R + T - xp - m'] \end{aligned} \tag{3}$$

First order conditions require $1/x = p/p_y$, $Zpf_\ell(\ell_e) = wR$. Euler equation for money holding requires $p'_y/p_y = \beta R$, which can be rewritten as $\beta R = (1+g)(1+\pi)$ where$1 + \pi = p'/p$. When plug in interior solution for y for workers we have similar first order conditions: $1/x = p/p_y$, $w/p_y = v_\ell(\ell_w)$ and same Euler equation for money holding.

So we know both types of agents will consume the same amount of market goods, thus market clearing for market good requires $x = Znf(\ell_e)$. The labor market clearing condition requires: $n\ell_e = (1-n)\ell_w$. Credit market clearing requires that $M = w(1-n)\ell_w$, or to say, all the money are channelled to the entrepreneurs who need to buy labor inputs.

Lastly, the central bank can change the growth rate of money supply, τ, by changing the size of lump sum transfer. We will be focusing on the steady state equilibrium. In equilib-

[7]This is true when we have interior solution for y, or to say, Y is large.

rium agents take prices as given and maximize their value function and markets clear.

In steady state, from credit market clearing condition we know $w'/w = M'/M = (1 + \tau)$. Since w/Zp is a constant, so $1 + \pi = p'/p = (1 + \tau)/(1 + g)$. Therefore the Euler equation can be rewritten as

$$\beta R = 1 + \tau \tag{4}$$

Lemma 1. *An increase of τ will reduce w/Zp.*

Proof: $\frac{w}{p_y} = \frac{w}{Zp}\frac{Zp}{p_y} = \frac{w}{Zp}\frac{Z}{x} = \frac{w}{Zp}\frac{1}{nf(\ell_e)}$. Since $\frac{w}{p_y} = v_\ell(\ell_w)$, so the labor market clearing condition can be rewritten as

$$(1-n)v_\ell^{-1}(\frac{w}{Zp}\frac{1}{nf(\ell_e)}) - n\ell_e = 0,$$

where $\ell_e = f_\ell^{-1}(wR/Zp)$. LHS is increasing in both w/Zp and R so an increase of τ will increase R and thus will reduce w/Zp.■

2.2 Endogenous Entrepreneurship

Now let us assume at the end of each period, agents can choose their occupation in the next period. Thus the two value functions can be written as:

$$V^e(m; M, Z) = \max_{x,y,\ell_e,m'}\{log(x) + y + \beta \max\{V^e(m'; M', Z'), V^w(m'; M', Z')\}\}$$

$$V^e(m; M, Z) = \max_{x,y,\ell_w,m'}\{log(x) + y - v(\ell_w) + \beta \max\{V^e(m'; M', Z'), V^w(m'; M', Z')\}\}$$

Here free entry of entrepreneurship would require the payoff from being an entrepreneur is the same as that of a worker. Thus we have

$$\frac{1}{p_y}[Zpf(\ell_e) - wR\ell_e] - [\frac{w}{p_y}\ell_w - v(\ell_w)] = 0 \tag{5}$$

Theorem 1. *If $\ell_w = 1$ and $v(\ell_w) = 0$, or to say, individual labor supply is completely inelastic, then an increase of τ will increase n.*

Proof: Now the free entry condition can be written as $Zpf(\ell_e) - wR\ell_e - w = 0$, or

$$f(\ell_e) - \frac{wR}{Zp}\ell_e - \frac{wR}{Zp}\frac{1}{R} = 0. \tag{6}$$

Treat wR/Zp and R as two unknowns. The LHS is decreasing in wR/Zp and increasing R. So wR/Zp will increase as R increases. From the labor market clearing condition: $f_l^{-1}(wR/Zp) = (1-n)/n$, we know n is increasing in wR/Zp. Finally, from (4) we know that τ will increase R. ■

Notice in this case, even if the individual labor supply is inelastic, the labor supply in the aggregate level is elastic. This is consistant with many empirical finding that labor supply in the micro level is inelastic yet it is much more elastic in the macro level.

Further, from Lemma 1, this effect is very similar to the Tobin Effect: higher inflation induces more agents to become entrepreneurs because inflation makes being a worker so much less attractive, just as inflation encourages accumulation of capital because it makes holding the alternative, money, less attractive.

Next consider the opposite extreme case: when labor supply is compeletely elastic.

Theorem 2. *If $v(\ell_w) = \ell_w$, or to say, individual labor supply is completely elastic, then an increase of τ will decrease n.*

Proof: When $v(\ell_w) = \ell_w$, workers do not have any surplus by supplying labor. Therefore the Free Entry condition becomes a zero profit condition for the entrepreneurs: $Zpf(\ell_e) - wR\ell_e = 0$, or

$$[f(\ell_e) - \ell_e\frac{wR}{Zp}] = k/Zp.$$

The LHS is an decreasing function of wR/Zp thus we must have wR/Zp fixed, thus ℓ_e is also a constant and w/Zp is decreasing in R. Next notice interior solution for ℓ_w requires that $w = p_y$. Therefore $1/x = p/p_y = p/w$. So we know $x = nZf(\ell_e) = w/p$, or $nf(\ell_e) = w/Zp$. Since ℓ_e is constant, a decrease of w/Zp will decrease n as well. Thus τ will increase R and thus decrease n. ■

Here the intuition is that inflation can be thought as a tax on holding money, when labor supply being perfect elastic the buyers (the entrepreneurs) bear all the cost of inflation tax. Thus inflation will induce more entrepreneurs to become workers. The intrepretation of these two theorems is of following: First, from these two opposite extreme case, we know that when individual labor supply is relatively inelastic, then inflation tend to increase entrepreneurship, and vice versa. Second, given that most empirical studies using micro data show small number of labor supply elasticity, the first theorem might be more relevent.

In the rest of the paper, I will consider three extensions of this basic model: (1) Endogenous Growth; (2) Imperfect Bank Credit; and (3) Heterogeneous Entrepreneurship.

3 Market Economy with Perfect Bank Credit

3.1 Endogenous Growth

With probability σ , an entrepreneur comes up with an innovative idea which gives him/her a productivity boost over the current aggregate productivity ($\eta = \bar{\eta}$). With probability $1 - \sigma$, the entrepreneur receive a mediocre idea and he or she uses the publically known technology ($\eta = 1$). The reason to assume η is iid across time is to capture the idea that when an entrepreneur gets a good idea, he gets a boost of productivity in the short run and then the idea gets into the public domain and everyone can use it in the long run. The iid assumption also gives analytical tractability together with quasilinear utility.

Next let me introduce the evolution of aggregate productivity. Let N be the total number of innovations in each period. Aggregate productivity in the next period is given by the following formula: $Z' = G(N)Z$. In general I assume $G'(N) > 0$. Here I give an example of the evolution of aggregate productivity:

$$Z' = \rho[N\bar{\eta}^{\varepsilon} + (1 - N)]^{1/\varepsilon} Z$$

where ρ is an exogenous component and ε is a parameter capturing the substitutability of individual innovations in generating aggregate knowledge. These innovations get aggregated into the public knowledge and every agent knows the technology Z' next period. As special cases: if $\epsilon = 1$, then aggregate productivity next period is just an average of the technology everyone knows this period, if $\epsilon = +\infty$, then it is given by the maximum productivity; and $\epsilon = -\infty$ implies it is given by minimum productivity. This formulation is the same as the one used in Chiu, Meh and Wright (2011). The difference is: there N is determined by the trade between innovators and entrepreneurs and here entrepreneurs get ideas themselves so N depends directly on the total number of entrepreneurs: since I assume each entrepreneur gets innovation with probability \sigma , so $N = n\sigma$. Therefore $G'(n) > 0$.

This formulation captures the idea that entrepreneurs seek innovations that generate short-term profits for themselves end up causing positive externalities to aggregate productivity. Using this setup, inflation can enhance growth if it can induce more people to become entrepreneurs.

3.2 Imperfect Bank Credit

In this section I will show how the model works with perfect bank credit, which means anyone can borrow as much as he/she wants. Moreover, I will take the number of entrepreneurs as fixed at n in this section. First look at the profit maximization problem faced by an entrepreneur with money holding m and realized idiosyncratic innovation shock η, while W^e is his end-of-period wealth from market activity:

$$W^e(m, \eta; M, Z) = \max_{\ell,d,b} \eta Z p f(\ell) + (d-b)R \tag{7}$$

$$s.t. \quad w\ell = m - d + b \tag{8}$$

$$d \leq m, \tag{9}$$

where M is the aggregate money stock at the beginning of the period; d and b are the entrepreneur's deposit and borrowing from the financial market. Constraint (8) says the entrepreneur needs to pay his/her production cost before realizing revenue and (9) simply requires the deposit is no greater than the initial money holding at the beginning of the period. Because with perfect bank credit there is no restriction on b, we can plug in $d - b$ from (8) into (7), then we have

$$W^e(m, \eta; M, Z) = \max_{\ell} \eta Z p f(\ell) + (m - w\ell)R. \tag{10}$$

So the first order condition for ℓ is

$$Z\eta p f_\ell(\ell) = wR. \tag{11}$$

The LHS is the marginal nominal benefit of purchasing an additional unit of labor input while the RHS is the marginal cost of purchasing an additional unit of labor input. The reason why the marginal cost is wR is because instead of paying an additional unit of input at the beginning of the period, entrepreneurs can deposit w in the financial market and get R at the end of the period.

Similarly, the end-of-period wealth from market activity for a worker is given by:

$$W^w(m, 1; M, Z) = w + mR, \tag{12}$$

where a worker does not receive any innovation shocks so the second argument is 1. This says that a worker's end-of-period wealth from market activity is just wage plus his/her principal and interest from deposit. This is because a worker does not need to use the money brought into the current period and will deposit all of it in the financial market. Notice (10) and (12) imply that

$$\partial W^i(m, \eta; M, Z)/\partial m = R. \tag{13}$$

Given the two wealth functions, the value function of an agent is given by

$$
\begin{aligned}
V^i(m,\eta;M,Z) &= \max_{x,y,h} log(x) + y + \beta E V^i(m',z';Z) \\
&s.t. \quad xp + yp_y + m' = ZYp_y + W^i(m,\eta;M,Z) + T,
\end{aligned} \tag{14}
$$

where i could either be e or w, standing for entrepreneur or worker respectively. T is the lump sum transfer or tax imposed by the government. Specifically we will let $T = \tau M$, where τ is the growth rate of the money supply. Solve for interior solutions for $y > 0$ and eliminate y, then

$$
\begin{aligned}
V^i(m,\eta;M,Z) &= \max_{x}\{log(x) - x\frac{p}{p_y}\} + ZY + \frac{1}{p_y}[W^i(m,\eta;M,Z) + \tau M] \\
&+ \max_{m'}\{-\frac{1}{p_y}m' + \beta E V^i(\eta',m';M',Z')\}.
\end{aligned} \tag{15}
$$

The first order condition for x is

$$
\frac{1}{x} = \frac{p}{p_y}. \tag{16}
$$

It is clear from (18) that consumption of the market good x does not depend on either the occupation or individual state variables of an agent[8]. Despite this, y could vary depending on their occupations and state variables. The Euler equation for money holding is

$$
\frac{1}{x}\frac{1}{p} = \beta\frac{1}{x'}\frac{1}{p'}W^i(m',\eta';M',Z')/\partial m'. \tag{17}
$$

The LHS is the marginal cost of bringing one unit of currency into the next period while the RHS is the marginal benefit. The term $W^i(m',\eta';M',Z')/\partial m'$ is the marginal benefit of one unit of currency to the end-of-period nominal wealth in the next period. Because of (13), this Euler equation does not depend on any individual information, so everyone is

[8]As in models following Lagos and Wright (2005), the quasilinear utility together with iid shocks simplify the analysis.

indifferent of holding any amount of money. Therefore the money holding distribution is indeterminate. Assume for simplicity that all entrepreneurs hold m_e and all workers hold m_w, then labor market clearing and credit market clearing requires that

$$n\sigma f_l^{-1}(\frac{wR}{\bar{\eta}Zp}) + n(1-\sigma)f_l^{-1}(\frac{wR}{Zp}) = 1 - n, \tag{18}$$

$$n\sigma[m_e - wf_l^{-1}(\frac{wR}{\bar{\eta}Zp})] + n(1-\sigma)[m_e - wf_l^{-1}(\frac{wR}{Zp})] + (1-n)m_w = 0. \tag{19}$$

Then, using the fact that $nm_e + (1-n)m_w = M$, we have that[9]

$$M = (1-n)w. \tag{20}$$

Lastly we need market clearing for the market good, that is

$$x = n\sigma Z\bar{\eta}f[f_l^{-1}(\frac{wR}{\bar{\eta}Zp})] + n(1-\sigma)Zf[f_l^{-1}(\frac{wR}{Zp})]. \tag{21}$$

We are interested in balanced growth equilibrium, which is defined as prices $\{p, p_y, w\}$, nominal interest rate $\{R\}$ and allocations $\{x, y, \ell, d, b, m'\}$ that satisfy (a) utility maximization; (b) profit maximization; (c) markets clearing; and (d) constant growth of aggregate state variables.

These conditions require that the following objects will be constant: $G(n)$, τ, $\frac{w}{Zp}$, R, and $\frac{x}{Z}$.

First, to solve for R, we will notice using $p_y = xp$, (20) and the fact that w/Zp is constant, we know $p_y'/p_y = (1+\tau)$, which is the gross growth rate of the money supply. Then, using (13), we can rewrite the Euler equation as[10]

$$(1+\tau)\frac{1}{\beta} = R \tag{22}$$

[9] Actually for any money holding distribution we will end up with (20).

[10] The Euler equation can also be written as $(1+g)(1+\pi) = \beta R$, where π is the inflation rate for market good x.

The LHS can be seen as the gross nominal interest rate of an illiquid bond sold at the end of a period that matures at the end of the next period, and R is the nominal interest rate of deposits/loans after agents realize their idiosyncratic shocks. When there is no commitment problem between banks and agents, these two interest rates are the same. If this is the case, even workers will be willing to hold money. In this case the money holding decision is arbitrary, and any person is happy to hold any amount. There is only one thing different for entrepreneurs and workers: entrepreneurs sometimes are borrowers, but workers are never borrowers. In this environment an increase in τ will increase R one-for-one.

The equilibrium is characterized by (18) and (22). The two unknowns in the system are R and w/Zp, or the real wage in terms of the market good adjusted for productivity. Once we get R from (18), because now n is fixed, we can find w/Zp from the inputs market clearing condition (18). Since (18) is decreasing in both R and w/Zp, we are sure that the equilibrium exists and is unique.

The main take-away from this version of the model is that the money holding distribution is indeterminate and with fixed n, an increase in τ translates to an one-to-one increase in R, and a one-to-one decrease in w/Zp.

3.3 On the Timing of the Environment

In this subsection I will discuss three assumptions about timing in the model: first, the timing of innovation shocks; second, the timing of the lump-sum transfer; third, the timing of debt repayment.

First, I assume that the iid innovation shocks happens at the beginning of each period. The reason for this assumption is because I am mainly interested in the short-term needs of liquidity of entrepreneurs. This captures the idea that when an entrepreneur needs liquidity (or more specifically money), he/she needs to work with whatever liquidity he/she has on hand plus whatever liquidity he/she can borrow from the credit market (or banks). This is exactly what makes the question interesting: because of the uncertainty of the innovation shocks, an entrepreneur will anticipate this liquidity need and choose his/her money holding accordingly.

Second, I assume that lump-sum transfers take place at the end of the period. Other possibilities include: (a) after agents access financial market and (b) at the beginning of the period before agents realize their innovation shocks and access the financial market. (a) is obviously not a good choice because once the transfer happens agents will immediately want to access the financial market again, effectively changing (a) into (b). (b) is not too sensible because this effectively forces workers to put the money they receive as lump-sum transfers in the financial market. In this case lump-sum transfers not only change the money stock and thus price levels, but also affect the credit market in an artificial way. This being said, (b) is worthwhile considering from a purely theoretical point of view.

Third, I assume debts are repaid within the period. We should notice m' is the money holdings for agents before they realize their innovation shocks and access the financial market in the next period. As long as m' is higher than dR, that is, as long as the (targeted) money holdings at the beginning of next period is higher than the principle and interest on current deposits, then whether the debt is repaid at the end of the period or at the beginning of the next period before innovation shocks are realized is irrelevant. For concreteness, suppose that an agent wants his money holding in period $t+1$ to be \$200 and is expecting a repayment of \$150 from the financial market. Then, whether the \$150 is paid at the end of period t or at the beginning of period $t+1$ is the same for him: he puts aside \$50 at the end of period t and ends up having \$200 at the beginning of period $t+1$.

4 Imperfect Bank Credit

4.1 Exogenous Entrepreneurship

Now we come back to look at the cases where bank credit is not perfect. Specifically, we will focus our attention on the case where sometimes entrepreneurs are going to be constrained. Now the end-of-period wealth from market activity for an entrepreneur is given by

$$W^e(m,\eta;M,Z) = \max_{\ell,d,b} \eta Zpf(\ell) + (d-b)R \quad (23)$$

$$s.t. \quad w\ell = m - d + b \quad (24)$$

$$bR \leq \bar{b}Zp \quad (25)$$

$$d \leq m \quad (26)$$

Plug in the constraint (24), the wealth function becomes

$$W^e(m,\eta;M,Z) = \max_{l,d,b} \eta Zpf(\frac{m-d+b}{w}) + (d-b)R \quad (27)$$

subject to (25) and (26). Now, solving for d and b, we get

$$\begin{cases} d = 0 \; and \; b = \frac{\bar{b}Zp}{R}, & if \; \eta Zpf_l(\frac{m}{w} + \frac{\bar{b}Zp}{wR}) > wR \\ d > 0, \; and \; b < \frac{\bar{b}Zp}{R}, & otherwise \end{cases} \quad (28)$$

An entrepreneur will borrow to his limit if $\eta Zpf_l(\frac{m}{w} + \frac{\bar{b}Zp}{wR}) > Rw$. If this is not the case, then we will have interior solutions for d and b then $d - b = m - wl$. The end-of-period wealth from market activity for workers is the same as given by (12). Now the Euler equation (17) for entrepreneurs becomes

$$\frac{1+\tau}{\beta} = \sigma\bar{\eta}\frac{Zp}{w} f_\ell(\frac{m}{w} + \frac{\bar{b}Zp}{wR}) + (1-\sigma)R \quad (29)$$

As before the LHS is the marginal cost of carrying one more unit of money into the period in nominal terms, which is increasing in the growth rate of the money supply and decreasing in the discount factor. The RHS is the marginal benefit of one additional unit of money in terms of end-of-period nominal wealth: with $(1-\sigma)$ probability an entrepreneur gets no innovation shock, so he is unconstrained and just deposits the money into the bank while with σ probability an entrepreneur receives an innovation shock so he is constrained and $\sigma\bar{\eta}\frac{Zp}{w}f(\frac{m}{w} + \frac{\bar{b}Zp}{wR})$ is the nominal benefit of an additional unit of money for a constrained entrepreneur.

Lemma W: Workers do not hold cash across periods if some entrepreneurs are constrained.

Proof: As long as some entrepreneurs are constrained, from the first order condition (28) we know $\bar{\eta}\frac{Zp}{w}f_\ell(\frac{m}{w}+\frac{\bar{b}Zp}{wR}) > R$, so from (29) we know $R < (1+\tau)/\beta$. On the other hand, for workers to be willing to hold cash into the next period, we will need $R = (1+\tau)/\beta$, which cannot be satisfied. QED.

This is different from the perfect banking credit case. Another way to interpret this result is by observing: here the nominal interest rate in the credit market is depressed by the financial frictions, that is, lower than what the nominal interest rate would be when we have perfect bank credit. For entrepreneurs, they sometimes are constrained and have higher marginal benefit higher than the nominal interest rate, so they are willing to hold money. But for workers, the benefit of holding money is just the nominal interest rate which is now not enough to compensate for the cost of carrying money across periods.

Another question is whether some financial institutions can take deposits at the end of a period and make loans to the entrepreneurs at the beginning of the next period after they receive their innovation shocks. This cannot happen either: because workers will only be willing to deposit at the end of a period if financial institutions offer an interest rate R_d that is equal to $(1+\tau)/\beta$. But then the credit market interest rate is only $R < R_d$. Therefore no financial intermediary will borrow at R_d and lend at R. To sum up, in this environment only entrepreneurs will carry money and they will deposit and borrow in the credit market depending on their periodic productivity shocks.

Next we look at the two market clearing conditions. Inputs market clearing requires:

$$n\sigma(\frac{m}{w}+\frac{\bar{b}Zp}{wR}) + n(1-\sigma)f_l^{-1}(\frac{wR}{Zp}) = 1-n. \tag{30}$$

Credit market clearing (divided both sides by w) requires:

$$(1-\sigma)[m - wf_l^{-1}(\frac{wR}{Zp})] = \sigma\frac{\bar{b}Zp}{R} \tag{31}$$

Proposition 1: With exogenous entrepreneurship, when the borrowing constraint $bR \leq \bar{b}Zp$ is binding for the entrepreneurs with innovation shocks, there exists an unique equilibrium with $dR/d\tau > 0$ and $d(w/Zp)d\tau < 0$.

Proof: To solve for equilibrium, we will notice from (30), when n is fixed, there exists a unique wR/Zp. Now rewrite the Euler equation in the following way:

$$\frac{1+\tau}{\beta} = R[\sigma\bar{\eta}\frac{Zp}{wR}f_{\ell}(\frac{m}{w} + \frac{\bar{b}Zp}{wR}) + (1-\sigma)]. \tag{32}$$

Because wR/Zp is constant and $m/w = (1-n)/n$ is also constant, therefore an increase in τ induces a one-to-one change in R. Because wR/Zp is constant, an increase in R means a decrease in w/Zp. QED.

The intuition for this result is that when τ increases, the cost of carrying money from period to period increases. This lowers the entrepreneurs' demand for labor inputs. In order to maintain labor market clearing, we will need w/Zp to decrease. When this happens, total output of the market good does not change. Inflation acts as a transfer between workers and entrepreneurs[11].

Corollary 1: With exogenous entrepreneurship, inflation transfers wealth from workers to entrepreneurs.

Proof: We can look at the total wealth, including the lump-sum transfer for agents. Because we have shown that consumption of the market goods is the same for all agents, from (14) we can find consumption of the endowment good for the agents with different occupations:

$$y = ZY - \frac{xp}{p_y} + \frac{1}{p_y}[-m' + W^i(m,\eta;M,Z) + \tau M]. \tag{33}$$

Since workers will set their $m' = 0$, and $W^w(m,\eta;M,Z) = w$. Now from (38), $xp = p_y$, so for workers

[11] This is of course because we held the labor supply in the formal sector constant. Otherwise inflation will decrease labor supply thus total output could change. I do not consider that here because if we put a disutility from working in the market for workers, then we need to say something about disutility from being an entrepreneur. There is no standard way to model entrepreneurs' disutility. This will be further discussed in the last section.

$$y_w = ZY - \frac{xp}{p_y} + \frac{Z}{x}\frac{w}{Zp} + \tau\frac{Z}{x}\frac{w}{Zp}\frac{M}{w}. \quad (34)$$

Because total production of market goods is unchanged with respect to τ, so Z/x is constant. The third term is workers' wage in terms of the market good adjusted for productivity, and the fourth term is the value of the government transfer in terms of the endowment good. Though we know w/Zp is decreasing in τ, but it is not straightforward to see whether the fourth term is increasing or decreasing in τ. But if we rewrite the expression:

$$y_w = ZY - \frac{xp}{p_y} + \frac{Z}{x}\frac{w}{Zp}n + \frac{Z}{x}\frac{wR}{Zp}\frac{1+\tau}{R}(1-n) \quad (35)$$

Because x, Z/x, wR/Zp and $(1+\tau)/R$ are constant, and w/Zp is decreasing in τ so we know y_w is decreasing in τ. We know that the total production of market goods and the total endowment good are unchanged. So inflation acts just as transfer from workers to entrepreneurs. QED.

4.2 Endogenous Entrepreneurship

In this subsection, I will allow agents to choose between two occupations: entrepreneurs and workers. Specifically, at the end of the period, each agent, regardless of current occupation, can choose for next period whether to become a worker or an entrepreneur. Now the agents' problem is given by

$$\begin{aligned} V^i(m,\eta;M,Z) &= \max_{x,y} log(x) + y + \max_{m'}\{\beta EV^e(m',\eta';M,Z),\ \beta V^w(m',1;M,Z)\} \\ s.t.\quad & xp + yp_y + m' = ZYp_y + W^i(m,\eta;M,Z) + \tau M. \end{aligned} \quad (36)$$

Solve for interior solutions of y

$$V^i(m,\eta;M,Z) = \max_x\{log(x) - x\frac{p}{p_y}\} + ZY + \frac{1}{p_y}[W^i(m,\eta;M,Z) + \tau M]$$
$$+\max_{m'}\{-\frac{1}{p_y}m' + \max\{\beta EV^e(m',\eta';M,Z),\ \beta V^w(m',1;M,Z)\}. \quad (37)$$

Since Lemma W still holds here, the optimal money holding for a worker is zero. Now consider the case when $n < \bar{n}$, or to say, there are still potential entrepreneurs that choose to be workers. Then these agents must be indifferent about either being a worker or an entrepreneur[12]. Then we must have

$$\beta V^w(m',1;M',Z') \leq \max_{m'}\{-\frac{1}{p_y}m' + \beta EV^e(m',\eta';M',Z')\}, \quad (38)$$

where the inequality is binding if $n < \bar{n}$. Since x' and $\tau M'$ are independent of state variable and occupational choices, we have the following free entry condition (in terms of the previous period):

$$-\frac{1}{p_{y,-1}}m + \beta\frac{1}{p_y}E_{-1}W^e(m,\eta;M,Z) \geq \beta\frac{1}{p_y}w, \quad (39)$$

where the inequality is binding if $n < \bar{n}$. The expected wealth for market activity can be written as

$$E_{-1}W^e(m,\eta;M,Z) = \sigma[pZ\bar{\eta}f(\ell_H) - \bar{b}Zp] + (1-\sigma)[pZf(\ell_L) + (m - w\ell_L)R], \quad (40)$$

where $\ell = \frac{1-n}{n} + \frac{\bar{b}Zp}{\theta wR}$ and $\ell_L = f_\ell^{-1}(\frac{wR}{Zp})$. Notice $\sigma\bar{b}Zp$ is the amount the constrained entrepreneurs (borrowers) need to pay back at the end of the period, and $(1-\sigma)(m - w\ell_L)R$ is the amount the unconstrained entrepreneurs (lenders) receive at the end of the period. Notice from credit market clearing (31) that we know that these two terms cancel out in (40). This is simply because borrowers' debts are the deposits of the lenders. After some algebra,

[12]One can add fixed costs or benefit of being an entrepreneur, which will not qualitatively change the analysis.

we arrive at the following simplified free-entry condition for entrepreneurship:

$$\sigma\bar{\eta}f(\ell_H) + (1-\sigma)f(\ell_L) - \frac{1+\tau}{\beta}\frac{1-n}{n}\frac{w}{Zp} \geq \frac{w}{Zp}, \tag{41}$$

where the inequality is binding if $n < \bar{n}$. The endogenous variables are R, w/Zp and n. Now the equilibrium is characterized by three conditions: (30), (32), and (41).

Proposition 2: With a binding borrowing constraint for entrepreneurs with innovation shocks and endogenous entrepreneurship, there exists a unique equilibrium with $dR/d\tau > 0$, $d(\frac{w}{Zp})/d\tau < 0$, and when $n < \bar{n}$ we have $dn/d\tau > 0$.

Proof: See appendix.

We can get some intuition for Proposition 2 by looking at the simplified free-entry condition for entrepreneurship. The RHS of (41) is what a worker gets from market activity (adjusted for productivity). The LHS of (41) is the expected benefit of being an entrepreneur from market activity (adjusted for productivity): the first two terms are the expected revenue in terms of market goods (adjusted for productivity), and the third term is the expected cost of being an entrepreneur. The cost is increasing in w/Zp, the real wage in terms of market goods adjusted for productivity. It is also increasing in $(1-n)/n$, the expected measure of labor inputs each entrepreneur hires. Lastly, and most interestingly, the cost is increasing in $(1+\tau)/\beta$. The reason for this is that entrepreneurs need to sacrifice consumption in the previous period to prepare the liquidity/money needed for their business in the current period. Such sacrifice is increased when we have a higher growth rate of the money supply.

When n is fixed, then changes inτ do not change the LHS, because as shown in Proposition 1, wR/Zp and $(1+\tau)/\beta R$ are constant. But an increase in τ will decrease the RHS, w/Zp. As a result, inflation distorts the payoffs from different occupations. The only way to equate the LHS and RHS is to increase n.

When $n = \bar{n}$, we need to show that an increase in τ would not induce the existing entrepreneurs to become workers. If we fix n, we know from Proposition 1 and Corollary 1 that inflation will shift wealth from workers to entrepreneurs, so any further increase in τ will not change n.

From Proposition 2, we know that an increase in the growth rate of the money supply can increase the number of entrepreneurs. Because of $G'(n) > 0$, economic growth can be enhanced by higher rate of growth of money supply. Also notice from Proposition 2, when $n = \bar{n}$, any further increase in the growth rate of the money supply will not affect economic growth. So this channel only works for low inflation rates.

5 Concluding Remarks

In this paper, I examine how inflation affects agents' occupational choices of whether to become an entrepreneur or a worker. Inflation is a tax on holding money which increases the cost of being an entrepreneur. Although this is true, it ends up being passed on to workers in the form of lower real wages. Once occupational choice is allowed, a higher inflation rate will induce more people to become entrepreneurs instead of workers. More entrepreneurs generate more innovation and thus higher growth. This mechanism is in line with the literature relating growth to entrepreneurship and is novel compared to the existing literature studying the effects of long-run inflation on economic growth. In addition, this mechanism could help explain why long-run inflation and growth can be positively correlated when inflation levels are low. I will discuss two possible extensions/limitations of the model in the rest of this section.

In the current version of the model, this mechanism will only be effective at low inflation rate because I assume that there is a fixed number of identical potential entrepreneurs. It is worthwhile to explore a more realistic case: what will happen if potential entrepreneurs differ in their abilities to come up with innovations? Future versions of this paper will address this question explicitly. Here I give an informal argument. Note that the way inflation drives more people into entrepreneurship, in this paper, is by lowering real wages. When inflation is low, wages for workers are still high. The lower ability potential entrepreneurs would rather work for wages and the marginal agent must have high ability. The effect of inflation on growth would therefore be big. As inflation increases and the real wage decreases, the ability of the marginal potential entrepreneur will decrease as well. So the effect of this channel will be decreasing in inflation rates. It is possible that when inflation is high other channels will dominate this channel so the net effect of inflation on growth could be negative.

Another potential issue concerns the inelastic labor supply assumed in the model. If workers supply labor inelastically, all the inflation tax is borne by the workers, while the expected payoff for an entrepreneur from market activity remains the same. In general, if individual labor supply is elasticity, then both entrepreneurs and workers will suffer from the inflation tax and which group suffers more depends on preferences and technologies. However, there exists a broad literature about hours constraints i.e. workers cannot easily adjust their hours worked. Hours constraints may comes from a fixed cost per worker faced by employers, implicit contracts, agency problems and firm-specific human capital. See Martinez-Granado (2005) for a survey of the theoretical and empirical studies that support the existence of hours constraint. Another simple reason for hours constraints not discussed in this literature could be that employers across countries are required to pay higher hourly wages for overtime work, so individual labor supply might be constrained from above when inflation is low and real wages are still high. According to this explanation, if enforcement of labor law is weaker in emerging economies and thus individual labor supply is not in its corner solution, then inflation would hurt entrepreneurs as much. Inflation in this case would not enhance growth as much as when we have corner solutions for individual labor supply. This might even help explain why Lopez-Villavicencio and Mignon (2011) did not find a positive correlation of inflation and economic growth in these economies.

Nevertheless, it is interesting to know the quantitative implications of elastic labor supply on the mechanism proposed in this paper. For example, it is unrealistic to think that workers are willing to supply the same labor when real wages are extremely low, which happens when inflation is extremely high in the model. Labor supply elasticity will be studied more in future versions of this paper.

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Appendix

Proof of Proposition 2:

There are three equations characterizing the equilibrium. Euler Equation for money holdings:

$$\Gamma^1(R, \frac{w}{Zp}, n) = \sigma\bar{\eta}\frac{Zp}{w} f_l(\frac{1-n}{n} + \frac{\bar{b}Zp}{wR}) + (1-\sigma)R - \frac{(1+\tau)}{\beta} = 0 \tag{42}$$

Inputs market clearing:

$$\Gamma^2(R, \frac{w}{Zp}, n) = \sigma\frac{\bar{b}Zp}{wR} + (1-\sigma)f_l^{-1}(\frac{wR}{Zp}) - (1-\sigma)\frac{1-n}{n} = 0 \tag{43}$$

Free entry for entrepreneurship:

$$\Gamma^3(R, \frac{w}{Zp}, n) = \sigma\bar{\eta} f(\frac{1-n}{n} + \frac{\bar{b}Zp}{wR}) + (1-\sigma)f[f_l^{-1}(\frac{wR}{Zp})] - \frac{1+\tau}{\beta}\frac{1-n}{n}\frac{w}{Zp} - \frac{w}{Zp} \geq 0 \tag{44}$$

To show existence and uniqueness. From (43), we know wR/Zp is increasing in n. Condition (42) can be rewritten as

$$\frac{(1+\tau)}{\beta}\frac{w}{Zp} = \sigma\bar{\eta} f_l(\frac{1-n}{n} + \frac{\bar{b}Zp}{wR}) + (1-\sigma)\frac{wR}{Zp}. \tag{45}$$

The RHS is increasing in n, there fore w/Zp is increasing in n as well. From (43) we have

$$f_l^{-1}(\frac{wR}{Zp}) = \frac{1-n}{n} - \frac{\sigma}{1-\sigma}\frac{\bar{b}Zp}{wR}.$$

Now the derivative w.r.t. n of Γ^3 becomes

$$\begin{aligned}\partial\Gamma^3/\partial n \quad = \quad & -\frac{1}{n^2}\sigma\bar{\eta} f_l(\frac{1-n}{n} + \frac{\bar{b}Zp}{wR}) + \sigma\bar{\eta} f_l(\frac{1-n}{n} + \frac{\bar{b}Zp}{wR})\partial\frac{\bar{b}Zp}{wR}/\partial n \\ & +(1-\sigma)f_l(\frac{1-n}{n} - \frac{\sigma}{1-\sigma}\frac{\bar{b}Zp}{wR})(-\frac{1}{n^2} - \frac{\sigma}{1-\sigma}\partial\frac{\bar{b}Zp}{wR}/\partial n) \\ & +\frac{1}{n^2}\frac{1+\tau}{\beta}\frac{w}{Zp} - \frac{1+\tau}{\beta}\frac{1-n}{n}\partial\frac{w}{Zp}/\partial n - \partial\frac{w}{Zp}/\partial n\end{aligned}$$

Using (45), we have

$$\begin{aligned}\partial\Gamma^3/\partial n &= \sigma[\bar\eta f_l(\frac{1-n}{n}+\frac{\bar bZp}{wR})-\frac{wR}{Zp}]\partial\frac{\bar bZp}{wR}/\partial n\\ &\quad -\frac{1+\tau}{\beta}\frac{1-n}{n}\partial\frac{w}{Zp}/\partial n-\partial\frac{w}{Zp}/\partial n\\ &< 0\end{aligned}$$

From 43 we know that if $n\to 0$, then $wR/Zp\to 0$. From 45 we know that if $n\to 0$, then $w/Zp\to 0$. The first three terms in 44 is the surplus of being entrepreneur, which is positive even if $n\to 0$ (the analogy of an environment with finite agent model is the case with one entrepreneur having monopoly power over production), while the last term in 44, w/Zp goes to zero, so we know $\lim_{n\to 0}\Gamma^3>0$. On the other hand, when $n\to 1$, the first two terms of 44 converge to zero, while the last two terms can be grouped together as

$$-(\frac{1+\tau}{\beta}\frac{1-n}{n}+1)\frac{w}{Zp},$$

where the terms in the bracket is positive and w/Zp converges to infinity. So we know $\lim_{n\to 1}\Gamma^3<0$. So if $\bar n=1$, we must have a unique equilibrium. If $\bar n<1$, when $\Gamma^3(\bar n)<0$, we must have an unique equilibrium with$n<\bar n$; when $\Gamma^3(\bar n)>0$, then we must have an unique equilibrium with $n=\bar n$.

Derivatives (when $n<\bar n$):

Total differentiation of (42), (43) and (44), we have the differential equations system:

$$\left\{\begin{array}{c}\frac{1}{\beta}d\tau\\ 0\\ \frac{1-n}{n}\frac{w}{Zp}\frac{1}{\beta}d\tau\end{array}\right\}=\left\{\begin{array}{ccc}R_1 & \frac{w}{Zp}_2 & n_7\\ R_3 & \frac{w}{Zp}_4 & n_8\\ R_5 & \frac{w}{Zp}_6 & n_9\end{array}\right\}\left\{\begin{array}{c}dR\\ d(\frac{w}{Zp})\\ dn\end{array}\right\}=Q\left\{\begin{array}{c}dR\\ d(\frac{w}{Zp})\\ dn\end{array}\right\},$$

where

$R_1=\Gamma^1_1=(1-\sigma)-\sigma\frac{Zp}{w}\bar\eta f_{ll}(\bar l)\frac{\bar bZp}{wR}\frac{1}{R}>0$

$\frac{w}{Zp}_2=\Gamma^1_2=-\sigma(\frac{Zp}{w})^2[\bar\eta f_l(\bar l)+\bar\eta f_{ll}(\bar l)\frac{\bar bZp}{wR}]=[-\sigma(\frac{Zp}{w})^2\bar\eta f_l(\bar l)-\sigma(\frac{Zp}{w})^2\bar\eta f_{ll}(\bar l)\frac{\bar bZp}{wR}]$

$R_3 = \Gamma_1^2 = -\sigma\frac{\bar{b}Zp}{wR}\frac{1}{R} + (1-\sigma)\frac{1}{f_{ll}(l)}\frac{w}{Zp} < 0$

$\frac{w}{Zp}_4 = \Gamma_2^2 = -\sigma\frac{\bar{b}Zp}{wR}\frac{Zp}{w} + (1-\sigma)\frac{1}{f_{ll}(l)}R < 0$

$R_5 = \Gamma_1^3 = -\sigma\bar{\eta}f_l(\bar{l})\frac{\bar{b}Zp}{wR}\frac{1}{R} + (1-\sigma)\frac{1}{f_{ll}(l)}\frac{w}{Zp}\frac{wR}{Zp} < 0$

$\frac{w}{Zp}_6 = \Gamma_2^3 = -\sigma\bar{\eta}f_l(\bar{l})\frac{\bar{b}Zp}{wR}\frac{Zp}{w} + (1-\sigma)\frac{1}{f_{ll}(l)}R^2\frac{w}{Zp} - 1 - \frac{1+\tau}{\beta}\frac{1-n}{n} < 0$

$n_7 = \Gamma_3^1 = -\sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})\frac{1}{n^2} > 0$

$n_8 = \Gamma_3^2 = (1-\sigma)\frac{1}{n^2} > 0$

$n_9 = \Gamma_3^3 = (1-\sigma)\frac{1}{n^2}\frac{wR}{Zp} > 0$

To show $\det Q > 0$**:**

$$\begin{aligned}
\det Q &= n_7[R_3\frac{w}{Zp}_6 - \frac{w}{Zp}_4 R_5] + n_8[R_5\frac{w}{Zp}_2 - \frac{w}{Zp}_6 R_1] + n_9[R_1\frac{w}{Zp}_4 - \frac{w}{Zp}_2 R_3] \\
&= \frac{1}{n^2}\{-\sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})[R_3\frac{w}{Zp}_6 - \frac{w}{Zp}_4 R_5] + (1-\sigma)[R_5\frac{w}{Zp}_2 - \frac{w}{Zp}_6 R_1] \\
&\quad +(1-\sigma)\frac{wR}{Zp}[R_1\frac{w}{Zp}_4 - \frac{w}{Zp}_2 R_3]\},
\end{aligned}$$

where

$$\begin{aligned}
R_3\frac{w}{Zp}_6 - \frac{w}{Zp}_4 R_5 &= \frac{1}{R}[\sigma\frac{\bar{b}Zp}{wR} - (1-\sigma)\frac{1}{f_{ll}(l)}\frac{wR}{Zp}] \\
&\quad +\frac{1}{R}[\sigma\frac{\bar{b}Zp}{wR} - (1-\sigma)\frac{1}{f_{ll}(l)}\frac{wR}{Zp}]\frac{1+\tau}{\beta}\frac{1-n}{n}
\end{aligned}$$

$$\begin{aligned}
R_5\frac{w}{Zp}_2 - \frac{w}{Zp}_6 R_1 &= \sigma\bar{\eta}f_l(\bar{l})\frac{\bar{b}Zp}{\theta wR}\frac{Zp}{w}\frac{1}{R}\frac{1+\tau}{\beta} - (1-\sigma)\frac{1}{f_{ll}(l)}\frac{wR}{Zp}]\frac{1+\tau}{\beta} \\
&\quad +[(1-\sigma) - \sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}\frac{1}{R}] \\
&\quad +[(1-\sigma) - \sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}\frac{1}{R}]\frac{1+\tau}{\beta}\frac{1-n}{n}
\end{aligned}$$

and

$$R_1\frac{w}{Zp}_4 - \frac{w}{Zp}_2 R_3 = \frac{1+\tau}{\beta}[(1-\sigma)\frac{1}{f_{ll}(l)} - \sigma\frac{\bar{b}Zp}{wR}\frac{Zp}{wR}].$$

Therefore

$$
\begin{aligned}
\det Q &= \frac{1}{n^2}\{-\sigma\frac{Zp}{wR}\bar{\eta}f_{ll}(\bar{l})[\sigma\frac{\bar{b}Zp}{wR}-(1-\sigma)\frac{1}{f_{ll}(l)}\frac{wR}{Zp}] \\
&-\sigma\frac{Zp}{wR}\bar{\eta}f_{ll}(\bar{l})[\sigma\frac{\bar{b}Zp}{wR}-(1-\sigma)\frac{1}{f_{ll}(l)}\frac{wR}{Zp}]\frac{1+\tau}{\beta}\frac{1-n}{n} \\
&+(1-\sigma)[(1-\sigma)-\sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}\frac{1}{R}] \\
&+(1-\sigma)[(1-\sigma)-\sigma\frac{Zp}{w}\bar{\eta}f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}\frac{1}{R}]\frac{1+\tau}{\beta}\frac{1-n}{n} \\
&+\sigma(1-\sigma)\frac{1+\tau}{\beta}\frac{\bar{b}Zp}{wR}[\bar{\eta}f_l(\bar{l})\frac{Zp}{wR}-1]\}
\end{aligned}
$$

Notice $\bar{\eta}f_l(\bar{l})\frac{Zp}{wR}-1>0$ is from (28). Now all of the terms are positive, therefore $\det Q > 0$.

To find the derivatives. First for $dR/d\tau$

$$
\begin{aligned}
dR/d\tau &= \frac{1}{\det Q}\frac{1}{\beta}\det\begin{Bmatrix} 1 & \frac{w}{Zp}_2 & n_7 \\ 0 & \frac{w}{Zp}_4 & n_8 \\ \frac{1-n}{n}\frac{w}{Zp} & \frac{w}{Zp}_6 & n_9 \end{Bmatrix} \\
&= \frac{1}{\det Q}\frac{1}{\beta}\{(n_9\frac{w}{Zp}_4-n_8\frac{w}{Zp}_6)+\frac{1-n}{n}\frac{w}{Zp}(\frac{w}{Zp}_2 n_8-\frac{w}{Zp}_4 n_7)\}
\end{aligned}
$$

where

$$
\begin{aligned}
n_9\frac{w}{Zp}_4-n_8\frac{w}{Zp}_6 &= (1-\sigma)\frac{1}{n^2}\frac{wR}{Zp}[-\sigma\frac{\bar{b}Zp}{wR}\frac{Zp}{w}+(1-\sigma)\frac{1}{f_{ll}(l)}R] \\
&-(1-\sigma)\frac{1}{n^2}[-\sigma\bar{\eta}f_l(\bar{l})\frac{\bar{b}Zp}{wR}\frac{Zp}{w} \\
&+(1-\sigma)\frac{1}{f_{ll}(l)}R^2\frac{w}{Zp}-1-\frac{1+\tau}{\beta}\frac{1-n}{n}] \\
&= (1-\sigma)\frac{1}{n^2}\sigma\frac{\bar{b}Zp}{wR}[\frac{Zp}{wR}\bar{\eta}f_l(\bar{l})-1]+(1-\sigma)\frac{1}{n^2}[1+\frac{1+\tau}{\beta}\frac{1-n}{n}]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{w}{Zp}_2 n_8 - \frac{w}{Zp}_4 n_7 &= [-\sigma(\frac{Zp}{w})^2\bar{\eta} f_l(\bar{l}) - \sigma(\frac{Zp}{w})^2\bar{\eta} f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}](1-\sigma)\frac{1}{n^2} \\
&\quad -[\sigma(\frac{Zp}{w})^2\bar{\eta} f_l(\bar{l}) + \sigma(\frac{Zp}{w})^2\bar{\eta} f_{ll}(\bar{l})\frac{\bar{b}Zp}{wR}]\sigma\frac{Zp}{w}\bar{\eta} f_{ll}(\bar{l})\frac{1}{n^2} \\
&= -\sigma(1-\sigma)(\frac{Zp}{w})^2\bar{\eta} f_l(\bar{l})\frac{1}{n^2} + \sigma(1-\sigma)\frac{\bar{\eta} f_{ll}(\bar{l})}{f_{ll}(l)}R\frac{Zp}{w}\frac{1}{n^2} \\
&\quad -\sigma\frac{\bar{b}Zp}{wR}\frac{Zp}{w}\frac{Zp}{w}\bar{\eta} f_{ll}(\bar{l})\frac{1}{n^2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
dR/d\tau &= \frac{1}{\det Q}\frac{1}{\beta}\{(1-\sigma)\frac{1-n}{n}\frac{1}{n^2}[\frac{1+\tau}{\beta} - \sigma\frac{Zp}{w}\bar{\eta} f_l(\bar{l})] \\
&\quad +(1-\sigma)\frac{1}{n^2}\sigma\frac{\bar{b}Zp}{wR}[\frac{Zp}{wR}\bar{\eta} f_l(\bar{l}) - 1] + (1-\sigma)\frac{1}{n^2} \\
&\quad +\sigma(1-\sigma)\frac{\bar{\eta} f_{ll}(\bar{l})}{f_{ll}(l)}R\frac{1}{n^2}\frac{1-n}{n} - \sigma\frac{\bar{b}Zp}{wR}\frac{Zp}{w}\bar{\eta} f_{ll}(\bar{l})\frac{1}{n^2}\frac{1-n}{n}\}
\end{aligned}
$$

Notice from Euler equation $\frac{1+\tau}{\beta} - \sigma\frac{Zp}{w}\bar{\eta} f_l(\bar{l}) = (1-\sigma)R > 0$. So every term is positive thus $dR/d\tau > 0$.

Second for $d(w/Zp)/d\tau$

$$
\begin{aligned}
d(w/Zp)/d\tau &= \frac{1}{\det Q}\frac{1}{\beta}\det\begin{Bmatrix} R_1 & 1 & n_7 \\ R_3 & 0 & n_8 \\ R_5 & \frac{1-n}{n}\frac{w}{Zp} & n_9 \end{Bmatrix} \\
&= \frac{1}{\det Q}\frac{1}{\beta}\{(n_8R_5 - R_3n_9) + \frac{1-n}{n}\frac{w}{Zp}(R_3n_7 - R_1n_8)\},
\end{aligned}
$$

where

$$
n_8R_5 - R_3n_9 = -\sigma(1-\sigma)\frac{1}{n^2}\frac{\bar{b}Zp}{wR}\frac{1}{R}[\bar{\eta} f_l(\bar{l}) - \frac{wR}{Zp}],
$$

and

$$
\begin{aligned}
R_3 n_7 - R_1 n_8 &= \sigma \frac{Zp}{w} \bar{\eta} f_{ll}(\bar{l}) \frac{\bar{b} Zp}{wR} \frac{1}{R} \frac{1}{n^2} - (1-\sigma)(1-\sigma)\frac{1}{n^2} \\
& -\sigma(1-\sigma)\frac{1}{n^2}\frac{\bar{\eta} f_{ll}(\bar{l})}{f_{ll}(l)}
\end{aligned}
$$

Since $n_8 R_5 - R_3 n_9 < 0$ (from (28)) and $R_3 n_7 - R_1 n_8 < 0$, so $d(w/Zp)/d\tau < 0$.

Lastly for $dn/d\tau$

$$
\begin{aligned}
dn/d\tau &= \frac{1}{\det Q_2} \frac{1}{\beta} \det \left\{ \begin{array}{ccc} R_1 & \frac{w}{Zp}_2 & 1 \\ R_3 & \frac{w}{Zp}_4 & 0 \\ R_5 & \frac{w}{Zp}_6 & \frac{1-n}{n}\frac{w}{Zp} \end{array} \right\} \\
&= \frac{1}{\det Q} \frac{1}{\beta} \{ \frac{1-n}{n} \frac{w}{Zp} [R_1 \frac{w}{Zp}_4 - \frac{w}{Zp}_2 R_3] + [R_3 \frac{w}{Zp}_6 - R_5 \frac{w}{Zp}_4]\}
\end{aligned}
$$

where both $R_1 \frac{w}{Zp}_4 - \frac{w}{Zp}_2 R_3$ and $R_3 \frac{w}{Zp}_6 - R_5 \frac{w}{Zp}_4$ have been calculated before. Therefore

$$
\begin{aligned}
dn/d\tau &= \frac{1}{\det Q} \frac{1}{\beta} \{ -\frac{1+\tau}{\beta} \frac{1}{R} [\sigma \frac{\bar{b} Zp}{wR} - (1-\sigma) \frac{1}{f_{ll}(l)} \frac{wR}{Zp}] \frac{1-n}{n} \\
& + \frac{1}{R} [\sigma \frac{\bar{b} Zp}{wR} - (1-\sigma) \frac{1}{f_{ll}(l)} \frac{wR}{Zp}] \\
& + \frac{1}{R} [\sigma \frac{\bar{b} Zp}{wR} - (1-\sigma) \frac{1}{f_{ll}(l)} \frac{wR}{Zp}] \frac{1+\tau}{\beta} \frac{1-n}{n} \} \\
&= \frac{1}{\det Q} \frac{1}{\beta} \frac{1}{R} [\sigma \frac{\bar{b} Zp}{wR} - (1-\sigma) \frac{1}{f_{ll}(l)} \frac{wR}{Zp}] > 0.
\end{aligned}
$$