Auctioning Social Surplus: First Best Bayesian Mechanism with Ex Post Individual Rationality*

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October 30, 2009

Abstract

This paper constructs a concrete mechanism/auction to explore the consequence of imposing the ex post participation constraint. The main findings are:

(1) In private good cases (symmetric or asymmetric), we can obtain ex post first best, ex post budget balance, at least interim incentive compatibility and ex post individual rationality (we call it ex post social effiicency), whenever the VCG mechanism runs expected surplus. And the mechanism generating an ex post monotonic payoff is generically unique (up to an ex ante side-pay). In addition, compared with standard auctions, our mechanism generates a risk-free revenue to the seller and ex post invidually rational payoff to the bidders.

(2) In a general preference case with externality, we show there exists an ex post socially efficient mechanism when the number of participants is sufficiently small ($n = 2$). And the choice of mechanism depends on whether the quantity is endogeneous or not.

(3) As an implication, we provide a general discussion on how divisibility, endowment distribution and preference affect the possibility of trade. For the negative result, we show a set of conditions for non-existence of an ex post socially efficient trade, such as either utility is linear or the lowest type agent's utility is independent of his endowment, which can be regarded as stronger version of no-trade theorem (Myerson-Satterthwaite, 1983). This proposition implies non-existence of an ex post socially efficient partner dissolving mechanism. For the positive side, we provide a sufficient condition for existence of an ex post socially efficient trade mechanism and show an explicit example.

JEL Code: D82

*The early title of this paper goes by "on an ex post socially efficient mechanism".

1 Introduction

Suppose a social planner wants to allocate resources efficiently ex post, and he also subjects to budget balance ex post. Basically, we have the following seven combinations of interest (See Table 1).

Table. 1 Combinations of Properties of Ex Post Efficient Mechanisms.

	Individual Rationality	Incentive Compatibilty	Budget Balance
Ex ante	(1)	(4)	N.A.
Interim	(2)	(5)*	N.A.
Ex Post	(3)*	(6)	(7)*

The well-known classic Vickery-Clark-Groves (VCG hereafter) mechanism can be either ex post individually rational or ex post budget balance, but cannot be both. Holmstrom (1977) provides a necessary and sufficient condition for ex post budget balance. Another well-known mechanism by Arrow (1979) and d'Aspremont Varet-Gared (1979) is ex post budget balance but might not be interim individually rational. The celebrated result by Myerson-Satterthewaite (1983) shows the impossibility of having combination (2), (5) and (7) in the above table, when the agent's preferences are linear and the seller owns the object initially. Several other authors provide some additional insights with slight change of assumptions in endowment or divisibility of the good, where conterveiling incentive matters (Crampton, Gibbons and Klemperer, 1987; McAfee, 1992, *among others*). In their story, combinations (2), (5) and (7) might be possible.

Recently, Ledyard and Palfrey (2006) has fully characterized the solution to an interim socially efficient mechnism, i.e., combination (2), (5) and (7). And Krishna and Perry (1998) has generalized the classic Vickery-Clark-Groves mechanism to discuss the sufficient and necessary condition for existence of interm socially efficient mechanism. Unfortunately, Ely and Cheung (2003) shows combination of (3), (6) and (7) is impossible.

This paper is considering the possibility of the combination (3), (5) and (7), star marked in table 1. Our interest on ex post individual rationality is motivated by the fact that, the participant may opt out of ex post once the ex post allocation can not give him higher utility than the outside reservation utility. The consequence of this option is subtle: because the participant expects that somebody will opt out, his strategy at the interim stage will change given that the designer is not a budget breaker. Therefore, the interim incentive compatible constraint will be different from the mechanism without an option to quit. We first show that the condition for a mechanism being robust against ex post option to quit needs to satisfy an ex post individual rationality constraint. Then we check the condition under which an ex post efficient allocation exists, satisfying incentive compatible (interim), budget balance (ex post) and individual rational (ex post). The importance and significance of the ex post IR constraint have been discussed by Dudek, Kim and Ledyard (1995). Recently, there has been increasing literature on exploring robustness of mechanism design, which requires ex post IR, even ex post incentive compatibility (Morries, 2003; Bergemann, S. Morris, 2005). Our strategy of proof is to construct a concrete mechanism or auction which has these good properties.

Our mechanism or auction has a very intuitive intepretation, and seems simple in terms of pragmatic use, which can be regarded as an auction to sell social surplus (if any). The game can be decomposed into two stages: (i) the first stage, all bidders compete for the right to own entitlement (license). The owner of entitlement can charge full consumer surplus to the other bidders, but in order to win the entitlement, he needs to pay a lump sum transfer to the others; (ii) the second stage is a trivial game, where the bidders are charged according to their first stage reports. By constructure, this mechanism is always ex post budget balance. We show it will be also ex post individual rational under some conditions, depending on the context. In private good cases, it will be ex post individual rational whenever VCG runs expected social surplus. However, in public good cases, it sensitively depends on the flexibility of total supply and the number of participants.

We also use this mechanism to explore the possibility of no trade (Myerson-Satterthewaite theorem) in a generalized environment, where divisibility of trade object, distribution of initial endownment and concavity of preference have receieved full consideration. We show trade could happen when either endownment is extreme or relatively symmetric when preference is concave. But if we impose the ex post IR constraint instead of its interim counterpart, even though the endownment is symmetric, no trade happens, in constrast to the existence of efficient partner dissolving mechanism (Clampton, Gibbons, Klemperer 1987). We provide a set of conditions for non-existence of ex post socially efficient trade mechnanisms.

The closest paper to ours is Dudek, Kim and Ledyard (1995) (DKL, hereafter), though our discovery is independent of theirs and the motivation is quite different. They propose an ex post individually rational mechanism to allocate a single unit private good among agents whose reservation utility is type-independent[1]. However, our paper proposes an explicit auction format for

[1] Their paper is also motivated by the importance of a transfer among agents.

privitization[2], both for private good and public good, either for endogeneous or exogeneous quantity and derives the necessary and sufficient condition for the existence of ex post individually rational mechanisms. In addition, we also discuss how flexibity of endowment and type-dependent reservation affect the existence of the ex post individual rationality.

The connections of our paper to standard auctions also can be seen in the following senses. In private good cases (whether or not the quantity of supply is fixed), compared with the standard auction, the present auction format generates a risk-free revenue (same as any efficient allocation) to the seller (Eso, 1999), and meets all bidders ex post individual rationality without any side-payment. We can explicitly characterize the bidding strategy under asymmetric distribution; as a contrast, finding the solutions to the standard asymmetric first price auction is very complicated. In public good cases, our auction connects with multi-unit auction or divisible good auction (Wilson 1979, Ausubel, 2004, Wang and Zender, 2001; *among others*), but there are several differences worth mentioning. First, the bidding strategy in our auction is much more tractable, so that we can work out the solution explicitly, for general quansi-concave preferences. As a consequence, we can check the revenue easily, while in a standard multi-unit auction, it is hard to invert the demand function and work out the formula of expected revenue in general quasi-concave utility form. Second, the seller can earn a risk-free revenue, which is the highest revenue among all efficient allocations, and the bidders' ex post individual rationalities are met when the number of bidders is two. Third, under symmetric situation, the allocation in our auction is always efficient, while the standard auction may not be (Ausubel, 2004).

The paper then is organized as follows. Section 2 describes the basic setting of the environment and associated solution concepts. Section 3 deals with private good and Section 4 deal with public good. In section 5, we propose a specific auction to implement the mechanisms we proposed in section 3 or 4. We finally summarize the findings in Section 5. Technical proofs are in the Appendices.

[2]The bidding function in our auction will be consistent with DKL's mechanism when either in the symmetric independent private case or the two-bidder asymmetric independent private case. We also discuss the relevance of choice of an auction format, especially in public good cases.

2 Preliminaries

2.1 Classical environment

Environment: There are n players. Each i's utility is $u_i = v_i(x_i, \theta_i) - m_i$, where $v_i(x_i, \theta_i)$ is utility from consumption of good x_i, and m_i is money payment[3], where θ_i is only known by player i, which is a random variable drawn from some set $\Theta_i \subset \mathbb{R}$, with cumulation distribution function (c.d.f) $F_i(.)$ and probability distribution function (p.d.f) $f_i(.)$, but the distribution is common knowledge. And we assume $v_i(.,.)$ is an increasing function of x_i and θ_i, and satisfies supermoduarity, i.e., $v_i(x_i, \theta_i) + v_i(x_i', \theta_i') \geq v_i(x_i', \theta_i) + v_i(x_i', \theta_i)$ for any x_i', x_i, θ_i', θ_i (this means, the preference appears an increasing difference). We use $\mathbf{v} = (v_1, v_2, ..., v_n)$ to denote the collection of all variables and use subcript $-i$ to indicate all individuals except i ($\boldsymbol{\theta}$ and $\mathbf{x}$ have the same treatment).

Suppose the cost of building $\mathbf{x}$ takes a general form $c(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is the bundle of the goods. We assume $c(\mathbf{x})$ is non-decreasing with any arguments, and at least for some i, it is strictly increasing. To capture the possible externality, we assume $c(x_1, x_2, ..., x_n)$ appears piecewise submodular, namely, $c(x_i, x_j', x_{-ij}) + c(x_i', x_j, x_{-ij}) \geq c(x_i, x_j, x_{-ij}) + c(x_i, x_j, x_{-ij})$. This cost function can be a pure public good case if $c(x_1, x_2, ..., x_n) = c(\max x_i)$ such as national security, or partially/fully excludible public good $c(x_1, x_2, ..., x_n) = c(\sum x_i)$. Particularly, we allow $c(\mathbf{x})$ to have an infinite marginal cost at some exogenous point, which corresponds to situations where the total endownment is given (resambling a pure exchange economy in text books).

Allocation Rule: Let $\mathbf{X} \subset \mathbb{R}^n$ be an arbitrary set of allocations (feasible), and let $\mathbf{x} : \Theta \to \mathbf{X}$ be the social choice rule. Throughout this paper, we are interested in the following allocation rule, which is called optimal if

$$\mathbf{x}^*(\boldsymbol{\theta}) \in \begin{cases} \arg\max_{\mathbf{x}} \sum v_i(x_i, \theta_i) - c(\mathbf{x}) \text{ given state } \boldsymbol{\theta}\text{, if quantity is endogenous} \\ \arg\max_{\mathbf{x}} \sum v_i(x_i, \theta_i) \text{ s.t. } c(\mathbf{x}) \leq \bar{c} \text{ given state } \boldsymbol{\theta}\text{, if quantity is exogenous} \end{cases}$$

Unless pointed out explicitly, $S(\boldsymbol{\theta})$ is used to denote the social surplus of either case, i.e. $S(\boldsymbol{\theta}) = \max_{\mathbf{x}} \sum v_i(x_i, \theta_i) - c(\mathbf{x})$ or $S(\boldsymbol{\theta}) = \max_{\mathbf{x}} \sum v_i(x_i, \theta_i)\ s.t. c(\mathbf{x}) \leq \bar{c}$. The following proposition is a standard result based on supermodularity.

[3] This preference can be regarded as a general form of common value. $v_i(x, \theta_i) = \mathbb{E}[u(x, S)/S = \theta_i]$, where S is some random variable common to all players, and θ_i is a private signal. Many public good consumptions have such a feature, such as hospital space, energy, public transportation and so on. This specification of utility function is general enough to cover the private value and public good situation, associated with a certain form of cost function. In a pure public good case, all $x_i = x$. One may generalize the utility function to $v_i(h_i(\mathbf{x}), \theta_i)$ with $h_i : \mathbb{R}^n \to \mathbb{R}$, as Mookherjee and Reichelstein (1989).

Proposition 1 *Assuming $v_i(x_i, \theta_i)$ appears an increasing difference, and $c(x)$ appears a decreasing difference, then (i) in the endogenous quantity case, the $S(\theta)$ is supmodular and $x_i^*(\theta_i, \theta_j, \theta_{-ij})$ is non-decreasing in θ_i and θ_j. (ii) In the exogeneous quantity case, $x_i^*(\theta_i, \theta_j, \theta_{-ij})$ is non-decreasing in θ_i; in addition if $\sum x_i = \bar{x}$ and $v(x_i, \theta_i)$ differentiable in x_i, then $S(\theta)$ is submodular and $x_i(\theta_i, \theta_j, \theta_{-ij})$ is non-increasing with θ_j. (**Proof see Appendix A1**)*

Incentive compatibility: There are two concepts of interest, interim incentive compatible (IIC) or ex post incentive compatible (EPIC). They are defined as follows:

Definition 1 *A direct mechanism $< x, M >$ with $x : \Theta \to X$ being allocation rule and $M : \Theta \to R^n$ being payment rule, is interim incentive compatible if,*

$$\begin{aligned} \textit{IC(Bayesian)} \quad &: \quad \mathbb{E}_{\theta_{-i}} v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta_{-i}} M_i(\theta_i, \theta_{-i}) \\ &\geq \quad \mathbb{E}_{\theta_{-i}} v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta_{-i}} M_i(\tilde{\theta}_i, \theta_{-i}) \ \forall \tilde{\theta}_i, \theta_i \in \Theta_i \end{aligned}$$

The above concept means truth-telling is a Bayesian Nash Equilibrium (B.N.E.) strategy at the interim stage. If we use ex post equilibrium as solution concept, the incentive compatibility will be stronger, as follows.

Definition 3.2: *A direct mechanism $< x, M >$ with $x : \Theta \to X$ being allocation rule and M : $\Theta \to R^n$ being payment rule, is ex post incentive compatible if,*

$$\begin{aligned} \text{IC (Ex post)} \quad &: \quad v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i}) \\ &\geq \quad v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - M_i(\tilde{\theta}_i, \theta_{-i}) \ \forall \tilde{\theta}_i, \theta_i \in \Theta_i \ \theta_{-i} \in \Theta_{-i} \end{aligned}$$

Individual Rationality: Paralleling IC constraint, the participation constraint also can be defined as interim *individual rationality*,

$$\text{IR (Interim): } \mathbb{E}_{\theta_{-i}}[v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i})] \geq \underline{u}_i(\theta_i), \ \forall \theta \in \Theta$$

or *ex post individual rationality*,

$$\text{IR (Ex post):} \quad v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i}) \geq \underline{u}_i(\theta_i), \ \forall \theta \in \Theta$$

where $\underline{u}_i(\theta_i)$ is reservation utility, usually normalized to be zero if it is type independent. We will discuss this later.

Budget Balance: Ex post budget balance is defined as,

$$\text{BB (Ex post): } \sum M_i(\theta_i, \theta_{-i}) = \begin{cases} c(\mathbf{x}) \text{ if endowment is endogenous} \\ 0 \text{ if endowment is exogenous} \end{cases}$$

It is acceptable that under some situations, budget balance is not a problem, like union negatiation (government may subsidize one of the parties), while in many other situations, budget balance ex post is a constraint hard to break through[4].

It is well known (Groves (1971), Clark (1973), Vickery (1961), Laffont and Green (1977) Holmstrom, 1979 *among others*) that VCG mechanism with payment rule:

$$M_i^V(\hat{\theta}_i, \hat{\theta}_{-i}) = S(\underline{\theta}_i, \hat{\theta}_{-i}) - \sum_{j \neq i} v_j(x_j^*(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j) + c(\mathbf{x}^*(\hat{\theta}_i, \hat{\theta}_{-i})) \tag{1}$$

will implememt the optimal allocation rule in terms of both ex post IC and ex post IR. However, it is also well known that VCG defined as (1) can not be budget balance ex post. Another seminal mechanism propsed by Arrow (1979) and d'Aspremont-Garad-Varet (1979), (AGV hereafter) is budget balance ex post by constructure, but it may not be individual rational, either interim or ex post. The following two lemmas state the positve and negative sides of the existence of desirable mechanisms.

Lemma 1 *(Krishna and Perry, 1998; Krishna, 2002) There exists an efficient, incentive compatible (interim) and individual rational (interim) mechanism that balances the budget if and only if the VCG mechanism results in an expected surplus.*

$$\mathbb{E}\Delta \equiv \sum \mathbb{E}_{\theta_{-i}} S(\underline{\theta}_i, \theta_{-i}) - (n-1)\mathbb{E}_{\boldsymbol{\theta}} S(\theta_i, \theta_{-i}) \geq \sum \underline{u}_i(\underline{\theta}_i)$$

where $\underline{\theta}_i = \arg\min_{\theta_i} E_{\theta_{-i}} S(\theta_i, \theta_{-i}) - \underline{u}_i(\theta_i)$.

Krishna and Perry (1998) generalizes the conclusion that, among all individual rational (interim) and incentive compatible (interim) mechanisms, VCG maximizes the payment. If VCG runs deficit, no other mechanism can be better off.

However the negative result appears if instead using a stronger soluction concept as EPIC and EPIR. The following lemma states the impossibility (Chung and Ely, 2003).

Lemma 2 *(Chung and Ely, 2003) If the utility is increasing difference, there does not exist an optimal allocation that is ex post IR, ex post IC and ex post budget balance.*

[4] To taking an extreme example, regarding all humen beings as pariticipants in a global mechanism, then no third part can subsidize people living on earth. Of course ex post budget constraint may be too strong, sometimes it can be relaxed to a feasible constraint (Palfery and Ledyard,1999), which is non-defict constraint, $\sum M_i(\theta_i, \theta_{-i}) \geq c(\mathbf{x})$, where money left over is burned.

The existing literature tells that if ex post BB is imposed, one has to compromise, yielding IR and IC to interim sense. Interim IC is understandable since it is the best play given the information set the agent achieves, but interim IR exproiates the participant's right to quit even if participation is indeed not profitable ex post. This paper attents to discover what happens if this right is entitled to the players, and then they can "vote by foot" at the last minute when all types are revealed. In order to incorparate this feature, we extend the classical Bayesian game to a sequential game.

2.2 Effectiveness of option to quit ex post

The game is modified as the following:

Game: (i) Designer anounces the allocation rule of game $< \mathbf{x}, \mathbf{M} >$ with $\mathbf{x}$ being the allocation rule and $\mathbf{M} : \Theta \to \mathbb{R}^n$ with $M = (M_1, M_2, ..., M_n)$ as the rule of monetary transfer when the report is $\hat{\boldsymbol{\theta}}$. (ii) Knowing his own type θ_i only, each decides to be in or out of the mechanism; if he chooses "in", he needs to report his type (simulatenously with all opponents); (iii) all reports $\hat{\boldsymbol{\theta}} \in \Theta$ are published and $< \mathbf{x}, \mathbf{M} >$ is proposed; (iv) given the mechanism, for individual i, if he accepts $(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), M_i(\hat{\theta}_i, \hat{\theta}_{-i}))$ then he is implemented by what he accepts, or he can leave permanently (without any penality), obtaining an outside option $\underline{u}_i(\theta_i)$; (iv) finally, an up-to-date mechanism $< \mathbf{x}, \mathbf{M} >^*$ is enforced among all individuals who have not left. The allocation and payment in $< \mathbf{x}, \mathbf{M} >^*$ might be different from the original mechanism $< \mathbf{x}, \mathbf{M} >$ because some payment and allocations might no longer be plausible due to some participants' quiting. We denote this game by $\Gamma(n, \mathbf{v}, \mathbf{x}, \mathbf{M}, \boldsymbol{\Theta})$.

Equilibrium: The equilibrium here not only requires each player to tell the truth as a Bayesian Nash Equilibrium, but also requires that the expectation operation at the interim stage should be based on ex post participation set $\Theta^* \subset \Theta$, where $\Theta^* = \times_i\{\text{all } \theta_i \in \Theta_i\text{: } v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i}) \geq \underline{u}_i(\theta_i)\}$. The complication here is that Θ^* is endogenized by rule of game $< \mathbf{x}, \mathbf{M} >$, and affects the enforceability of $< \mathbf{x}, \mathbf{M} >$ in turn. For example, if a proposed mechanism results in $\Theta^* = \Phi$, then $< \mathbf{x}, \mathbf{M} >$ loses any power to be effective.

Implementability: A direct mechanism $< \mathbf{x}, \mathbf{M}; \boldsymbol{\Theta}^* >$ such that $\mathbf{x} : \Theta^* \to \mathbb{R}^n$ and $\mathbf{M} : \Theta^* \to \mathbb{R}^n$ with $\Theta^* \subset \boldsymbol{\Theta}$ is said to be implementable if

(i) for any $i \in \mathcal{N}^* \equiv |\boldsymbol{\Theta}^*|$, truth-telling is a BNE, i.e.

$$\begin{aligned} \text{IC(Bayesian)} \quad : \quad & \mathbb{E}_{\theta^*_{-i}} v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta^*_{-i}} M_i(\theta_i, \theta_{-i}) \\ \geq \quad & \mathbb{E}_{\theta^*_{-i}} v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta^*_{-i}} M_i(\tilde{\theta}_i, \theta_{-i}) \ \forall \tilde{\theta}_i, \theta_i \in \Theta^*_i \end{aligned}$$

(ii) Participation set is consistent, i.e.,

$$v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i}) \geq \underline{u}_i(\theta_i) \text{ for } \forall(\theta_i, \theta_{-i}) \in \Theta_i^*$$

Under this setting, a lot of mechanisms are no longer implementable even though they are initially implementable without requirement (ii). For example, AGV is one of the mechanism that fails (ii), as example 1 shows.

Regarding the complication of the above equilibrium, we particularly are interested in a typical implementable mechanism that all participants will not leave ex post, i.e., $\boldsymbol{\Theta}^* = \Theta$. This is a *full participation* mechanism. Therefore, the linkage between the current setting and classical mechaism design literature is obvious, throung the following theorem.

Theorem 1 *A direct mechanism* $< x, M; \Theta^* >$ *is full participation B.N.E. implementable if and only if (i)* $< x, M; \Theta >$ *is intereim incentive compatible, i.e.,*

$$\begin{aligned} \textit{IC (Bayesian)} \quad : \quad & \mathbb{E}_{\theta_{-i}} v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta_{-i}} M_i(\theta_i, \theta_{-i}) \\ \geq \quad & \mathbb{E}_{\theta_{-i}} v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta_{-i}} M_i(\tilde{\theta}_i, \theta_{-i}) \ \forall \tilde{\theta}_i, \theta_i \in \Theta_i \end{aligned}$$

and (ii) each individual's ex post IR constraint is met, i.e.,

$$\textit{IR (Ex post):} \qquad v_i(x_i(\theta_i, \theta_{-i}), \theta_i) - M_i(\theta_i, \theta_{-i}) \geq \underline{u}_i(\theta_i), \ \forall \theta \in \Theta$$

Proof. If part: use backward induction, if ex post IR is met, based on information set $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then not leaving is a best response to other players, given other individuals' not leaving. Given this future best response, back to the interim stage, IC constraint is consistent with support condition over $\boldsymbol{\Theta}^* = \Theta$; therefore, truth-telling is a B.N.E based on the information set θ_i. Therefore, $< \mathbf{x}, \mathbf{M}; \boldsymbol{\Theta} >$ is implementable.

Only if: If ex post IR condition is not met, at least some players are leaving, then it is not full participation implementable. Q.E.D. ■

Based on the above theorem, giving the optional right to the participant ex post can be thought of as putting the addtional ex post IR constraint on the program. For conceptual convenience, we define the ex post socially efficient mechanism below:

Definition 2 *A direct mechanism* (x, M) *is called ex post soically efficient if it maximizes "social surplus"* $(x \in x^*)$ *and at the same time satisfies IC (Bayesian), IR (Ex post) and BB (Ex post).*

The natural question arises here is, how significant is the difference that this extra constraint brings in? Intuitively, participants under ex post individually rational mechanisms, seem to have higher expected utility than under interim IR since they can always leave for higher payoff by voting by foot. Therefore, in order to "bribe" the players not to leave in equilibrium, the designer seemingly needs to have more surplus. In other words, is the expect surplus of VCG enough for such a kind of mechanism design?

2.3 Characterization of IC and Budget Balance Mechanism

Before proceeding, we characterize the incentive compatible condition first, which is a standard result in the existing literature. Let

$$m_i(z_i) = \int_{\theta_{-i}} M_i(z_i, \theta_{-i}) f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i}$$

be the expected payment when individual i reports z_i, therefore, in the social allocation game, for individual θ_i, his expected utility is

$$U_i(z_i, \theta_i) = \int_{\theta_{-i}} v_i(x_i^*(z_i, \theta_{-i}), \theta_i) f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i} - m_i(z_i) \tag{2}$$

It is well known that (Myerson, 1979; Laffort and Green, 1978; *among others*), a direct mechanism is incentive compatible, if and only if,

$$m'(\theta_i) = \left[\frac{\partial}{\partial z_i} \int_{\theta_{-i}} v_i(x_i^*(z_i, \theta_{-i}), \theta_i) f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i} \right]_{z_i = \theta_i} \geq 0 \tag{3}$$

Remark 1 *$x_i^*(z_i, \theta_{-i})$ needs not to be differentiable, but we assume that $\int_{\theta_{-i}} v_i(x_i^*(z_i, \theta_{-i}), \theta_i) f_{-i}(\theta_{-i}) d\theta_{-i}$ is differentiable in z_i. Of course, assuming $v_i(x_i^*(z_i, \theta_{-i}), \theta_i)$ is continuously differentiable function in (x, θ) is conventional.*

For the ex post budget balance mechnisms, the following lemma states the link between lemma 1 and the lowest type's payoff in the class of budget balance mechanism.

Lemma 3 *For any incentive compatible and ex post budget balance mechanism, the following inequality must be true:*

$$\sum U_i(\underline{\theta}_i) \leq \mathbb{E}\Delta$$

where $U_i(\underline{\theta}_i)$ is the lowest type's expected payoff under that mechanism. ***(Proof see Appendix A2)***

The above lemma builds an interesting linkage between budget balance mechanism and individually rational mechanism. Particularly, if $\underline{u}_i(\underline{\theta}_i) = 0$, it had better hold $\sum U_i(\underline{\theta}_i) = \mathbb{E}\Delta$, since individual rationality condition $\sum U_i(\underline{\theta}_i) \geq 0$ holds if and only $\mathbb{E}\Delta \geq 0$. For the typical approach to construct a budget balance mechanism, the crutial issue is to construct $m_i(\theta_i)$. Note that under any budget balance mechanism, there must be somebody who pays and others who are paid. So the allocation of payee/payer-ship is a key instrument, like conterveiling incentive. Our constructure of the mechanism is based on this intrinsic property of budget balance mechanism.

2.4 Related literature

1. Mechanism design and public good provision (Vickery, 1961; Groves, 1973; Clark, 1971), Green and Laffont (1977, 1979), Holmstrom (1977), Holmstrom and Myerson (1983). There is a lot of literature in this field.

2. Auction of shares (Wilson, 1979 QJE; Ausubel, 2004, AER among other): auction degign for divisible good quantity. Conclusion: no efficient allocation in general.

3. Partnership dissolving (Crampton, Gibbons and Klemperer, 1987, EMA, MacAfee, 1992, JET, Modouano *with others*, 2002;). There is efficient resolution in symmetric independent environment if the initial endownment is very symmetric. But not in general.

4. Ex post implementation and ex post mechanim design. Bergermann and Morries (2003), Chung and Ely (2003). In general, it is impossible to have an efficient mechanism satisfying ex post IC, IR and BB.

3 Private Good Senario

3.1 Mechanism under SIPV environment

In the case without externality, neither in utility nor in production, for individual i, his utility from consumption is $v_i(x_i, \theta_i)$ over x_i, and no one else benefits from x_i at all (this mean the only gets utility from his own consumption). We assume that θ_i is independent crossing individuals, and we will discuss both identical distribution or asymmetric distribution later. The cost function is $c(\max_i x_i)$. In this case,

$$S_i(\theta^{n:n}) = \max_{x_i} v_i(\theta^{n:n}, x_i) - c_i(x_i)$$

where we use $\theta^{i:j}$ to denote the i-th smallest order statisics among j random variables. It is reasonable to assume $S_i(\underline{\theta}_i) \geq 0$, implying that production is always socially efficient, otherwise, the socially efficient decision might be no production since outside option is higher[5].

In this case, VCG payment is the following:

$$M_i^V(\hat{\boldsymbol{\theta}}) = \begin{cases} S(\hat{\theta}_i) + c(x^*(\hat{\theta}_i)) & \text{if } \max_{j\neq i}\hat{\theta}_j < \hat{\theta}_i \\ \frac{1}{\#\mathcal{N}_j}[S(\theta^{n-1:n}) + c(x^*(\theta_i))] & \text{for any } \hat{\theta}_i \in \mathcal{N}_j \\ 0 & \text{if } \max_{j\neq i}\hat{\theta}_j > \hat{\theta}_i \end{cases}$$

VCG runs expected surplus if and only if $\mathbb{E}S(\theta)^{n-1:n} \geq 0$, which is always the case since $S_i(\underline{\theta}_i) \geq 0$. The question here is: can we allocate this expected surplus properly so that a budget balance mechanism is also ex post IR?

We propose the following mechanism to answer the above question.

M1: *(i) The highest type agent makes production decision by himself (therefore it is optimal); (ii) based on the report $\hat{\boldsymbol{\theta}}$, the payment rule associated with this allocation is:*

$$M_i^F(\hat{\boldsymbol{\theta}}) = \begin{cases} \frac{n-1}{n}\mathbb{E}[S(\tau^{n:n})/\tau^{n:n} \leq \hat{\theta}_i] & \text{if } \max_{j\neq i}\hat{\theta}_j < \hat{\theta}_i \\ \frac{1}{\#\mathcal{N}_j}\frac{n-1}{n}\mathbb{E}[S(\tau^{n:n})/\tau^{n:n} \leq \hat{\theta}_i] & \text{for any } \hat{\theta}_i \in \mathcal{N}_j \\ -\frac{1}{n}\mathbb{E}[S(\tau^{n:n})/\tau^{n:n} \leq \hat{\theta}_j^{n:n}] & \text{if } \max_{j\neq i}\hat{\theta}_j > \hat{\theta}_i \end{cases} \tag{4}$$

Under this decentralized mechanism, individual i's payoff is

$$u_i(\hat{\theta}_i, \theta_{-i}; \theta_i) = \mathbf{1}(\max_{j\neq i}\hat{\theta}_j < \hat{\theta}_i)S(\theta_i) - M_i^F(\hat{\boldsymbol{\theta}})$$

where $\mathbf{1}(Z)$ is indication function for event Z.

Remark 2 *This mechanism decentralizes the production decision, comparing with VCG, where the production decision is centralized. There is a centralized counterpart of M1, where the designer determines $x^*(\theta)$ and sells to the highest type reporter. The agents' behavior is equavalent (in term of strategy) under both versions. This conclusion holds even in the environment of externality.*

We claim this mechanism **M1** or $< \mathbf{x}^*(\boldsymbol{\theta}), \mathbf{M}^F(\boldsymbol{\theta}) >$ is an ex post socially effecient mechanism if and only if VCG runs expected social surplus, under SIPV environment.

Theorem 2 *Under SIPV, the proposed mechanism M1: $< x^*(\theta), M^F(\theta) >$ is ex post soically efficient if and only if VCG mechanim runs expected surplus.*

[5] If $c(x)$ is continuous, the first best solution x^{1st} solves $\frac{\partial}{\partial x}v(x, \theta^{n:n}) = c'(x)$, If x is not continous, like a binary variable, $S(\theta^{n:n})$ is still well-defined.

Proof. (i) To check the incentive compatibility, note that,

$$\begin{aligned} m(\theta) &= G(\theta)\left[\frac{n-1}{n}\frac{\int_{\underline{\theta}}^{\theta} S(\tau)dF(\tau))^n}{F(\theta)^n}\right] - \frac{1}{n-1}\int_{\theta}^{\bar{\theta}} \frac{n-1}{n}\left(\frac{\int_{\underline{\theta}}^{z} S(\tau)dF(\tau))^n}{F(z)^n}\right) dG(z) \\ &= (n-1)\int_{\underline{\theta}}^{\bar{\theta}} S(\tau)F(\tau))^{n-1}dF(\tau) - (n-1)\int_{\theta}^{\bar{\theta}} S(\tau)F(\tau))^{n-2}dF(\tau) \end{aligned}$$

therefore,

$$m'(\theta) = g(\theta)S(\theta)$$

which accords with the necessary and sufficient condition of interim IC.

(ii) Ex post budget balance is met by construction.

(iii) We check interim IR first, since if interim IR fails, the game can not proceed. It is easy to chek that

$$U(\theta) = G(\theta)S(\theta) - m(\theta)$$

therefore,

$$U(\underline{\theta}) = -m(\underline{\theta}) = (n-1)\int_{\underline{\theta}}^{\bar{\theta}} S(\tau)(1-F(\tau))F(\tau))^{n-2}dF(\tau) = \frac{1}{n}\mathbb{E}\Delta$$

So if and only if $\mathbb{E}\Delta \geq 0$, at the interim stage, nobody will quit. And at the ex post stage, if $\max_{j\neq i}\theta_j > \theta_i$, his ex post payoff $u^l(\theta) = \frac{1}{n}\mathbb{E}[S(\tau^{n:n})/\tau^{n:n} \leq \theta_j^{n:n}] \geq 0$; if $\max_{j\neq i}\theta_j < \theta_i$,

$$u^w(\theta) = S(\theta) - \frac{(n-1)}{n}\frac{\int_{\underline{\theta}}^{\theta} S(\tau)dF(\tau))^n}{F(\theta)^n} \geq u^l(\theta) > 0$$

This complete the proof of argument. Q.E.D. ■

It is also seen that M1 satisfies ex post monotonicity, defined as follows.

Definition 3 *A mechanism satisfies interim (ex post) payoff monotonicity if the interim (ex post) payoff is non-decreasing given any realization of other bidders' type.*

It is clear that IC interim requires interim payoff monotonicity, but not ex post monotonicity. And ex post monotonicity at least implies that in the ex post stage, the higher type can at least never be worse than the lower type. The existing mechanisms such as AGV do not satisfy ex post payoff monotonicity (as we will see an example later). Ex post payoff monotonicity is closely related to ex post incentive compatibility, because ex post payoff monotonicity is necessary for IC ex post but not sufficient. Meanwhile, ex post payoff monotonicity may imply that with some subset of realizations of state of the world, ex post IC is met.

Proposition 2 *Under SIPV, M1 satisfies ex post payoff monotonicity.*

Proof. Given any realization of other players' type, if individual i's type is lower than the winner's, say $\theta_i < \theta^{n:n}$, then his ex post payoff is independent of his type; if he himself is the highest type, $\theta_i = \theta^{n:n}$, then his ex post pay off is increasing of his type since $S(\theta) - \frac{(n-1)}{n}\frac{\int_{\underline{\theta}}^{\theta} S(\tau)dF(\tau))^n}{F(\theta)^n}$ is increasing with θ. The only trick here is to show that there is no drop when his type passes the pivotal point $\theta^{n-1:n}$. Note that $u^w(\theta) > u^l(\theta)$, so when passing the pivotal point, his payoff jumps rather than drops. And at the point $\theta_i = \theta^{n-1:n}$, the tie-setting makes his payoff in between $u^w(\theta)$ and $u^l(\theta)$, therefore, his expost payoff is non-decreasing given any realization of other bidders' type. Q.E.D. ■

Although M1 is not ex post incentive compatible, the ex post payoff monotonicity still generates several notable robust properties, in terms of ex post implementation. The agents will not regret the allocation under M1: the winner definitely does not want to be a loser by lowering his report; and the loser probably does not want to increase his report to become a winner under some subset of realizations as well. It seems acceptable that the winner can not renege on his payment, unless he wants to give up his winner-ship. Formally, we define winner's no veto power as below.

Definition 4 *Winner has no veto power in payment (WNVPP), if he cannot underpay without giving up his winner-ship.*

The above assumption rules out the winner's default such as simply paying less when he obtains the object. But we still allow the winner to change his report if he wants to switch the allocation. We have the following proposition.

Proposition 3 *Under SIPV, M1 with WNVPP is ex post incentive compatible with probability,*

$$\Pr(EPIC) = n\int_{\underline{S}}^{\bar{S}} \left[F(S^{-1}(\frac{\int_{\underline{\theta}}^{\theta} S(\tau)dF(\tau))^n}{F(\theta)^n})) \right]^{n-1} dF(S^{-1}(\tau)).$$

especially, when $F(S^{-1}(\tau)) = S^{\gamma}$, a power function, then

$$\Pr(EPIC) = (\frac{n\gamma}{n\gamma+1})^{(n-1)\gamma} \rightarrow \frac{1}{e} \cong 0.37$$

Proof. Given his oppenent's truth-telling ex post, the winner will not report $\tilde{\theta} > \theta$ ex post. And with WNVPP, the winner can not report $\tilde{\theta} < \theta$, but still holds the object. Moreover, the winner does not want to report $\tilde{\theta} \leq \theta_{n-1:n}$ too, since he earns less by changing allocation. Given the winner's behavior, the losers do not benefit by reporting $\tilde{\theta} < \theta$. The loser may only mispresent his type $\tilde{\theta} > \theta$ when the second highest type agent's realization is close enough to the winner's,

$$S_{n-1:n} - \frac{n-1}{n} \frac{\int_{\underline{\theta}}^{\theta_{n:n}} S(\tau) dF(\tau))^n}{F(\theta_{n:n})^n} \geq \frac{1}{n-1} \frac{\int_{\underline{\theta}}^{\theta_{n:n}} S(\tau) dF(\tau))^n}{F(\theta_{n:n})^n}$$

This event happens with probability

$$\Pr(S_{n-1:n} \geq \frac{\int_{\underline{\theta}}^{\theta_{n:n}} S(\tau) dF(\tau))^n}{F(\theta_{n:n})^n}) = 1 - n \int_{\underline{S}}^{\bar{S}} \left[F(S^{-1}(\frac{\int_{\underline{\theta}}^{\theta} S(\tau) dF(\tau))^n}{F(\theta)^n})) \right]^{n-1} dF(S^{-1}(\tau))$$

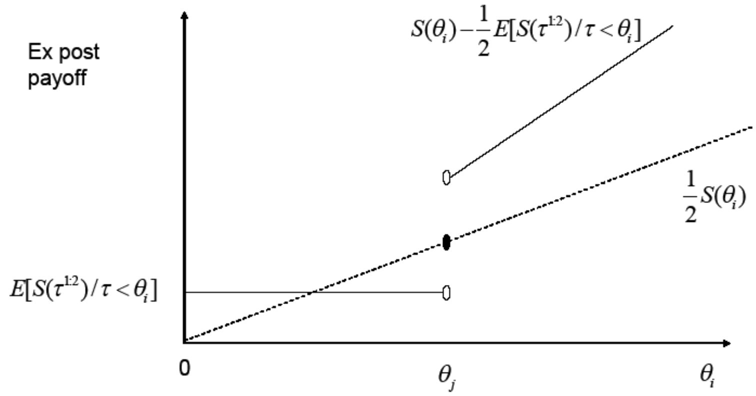
Thus, we have $\Pr(\text{EPIC}) = 1 - \Pr(S_{n-1:n} \geq \frac{\int_{\underline{\theta}}^{\theta_{n:n}} S(\tau) dF(\tau))^n}{F(\theta_{n:n})^n})$. It is easy to check when $F(S^{-1}(\tau)) = S^{\gamma}$, $\frac{\int_{\underline{\theta}}^{\theta} S(\tau) dF(\tau))^n}{F(\theta)^n} = \frac{n\gamma}{n\gamma+1}$, and $\Pr(\text{EPIC}) = (\frac{n\gamma}{n\gamma+1})^{(n-1)\gamma}$ follows. Q.E.D. ■

The above proposition indicates that if an interim incentive compatible mechanism wants to maximize the probability of ex post incentive compatibility, then ex post payoff monotonicity is required[6]. For example, the probability for AGV to be EPIC is zero even with WNVPP. We are wondering if other mechanisms also have nice properties as M1. The answer is no. M1 is the generically unique mechanism that satisfies ex post monotonicity and ex post individual rationality among all budget balance Bayesian mechanisms.

Theorem 3 *Under SIPV, M1 is the (generically) unique symmetric mechanism that is ex post socially efficient and satisfies ex post monotonicity (maximize probability of ex post IC). (Proof see A3)*

We compare M1 and AGV below. Take $n = 2$ as an example, the ex post pay-off structure of **M1** is indicated by Figure 1, which is a non-decreasing function. In a concrete example below, probability of winner being worse than loser in AGV is slightly greater than 60%, and probability not ex post IR is about 28%. The probability for **M1** to be EPIC is $\frac{\sqrt{3}}{3}$.

[6] The implication of the above proposition can also be understood as follows. If the designer only accepts the appeal that is pivotal (dismisses any ex post change of report if it is not pivotal), then with some probability, M1 is ex post incentive compatible.



(Insert Figure 1 here)

Example 1 ***Comparison between AGV and M1.***

Assume $n = 2$, θ *uniformly distributed over* $[0, 1]$, $c(x) = \frac{1}{\rho}\theta^\rho$, $v = x\theta$. *It is easy to see the first best social welfare is* $S(\theta) = \frac{1}{\gamma}\theta^\gamma$ *where* $\gamma \equiv \frac{\rho}{\rho-1}$.

Payoff Under AGV.

If i wins, his expost payoff is:

$$S(\theta_i) - \frac{\int_{\theta_j}^{\bar{\theta}} S(\theta)dF(\theta)}{1 - F(\theta_j)} = \frac{1}{\gamma(\gamma+1)}[\frac{\theta_i^\gamma(\gamma+1)(1-\theta_j) - (1-\theta_j^{\gamma+1})}{(1-\theta_j)}$$

Because when θ_j *is close to* θ_i *enough, since* $\gamma > 1$, *we have*

$$\theta_i^\gamma(\gamma+1)(1-\theta_i) - (1-\theta_i^{\gamma+1}) = \gamma\theta_i^\gamma(1-\theta_i) - (1-\theta_i^\gamma) < 0$$

So if the two agent's types are close enough, the winner may suffer even though he wins the object. For example, if $\gamma = 2$, *the cost is quadratic, then,*

$$\theta_j^2 < \theta_i^2 < \frac{1+\theta_j+\theta_j^2}{3}$$

is possible. The possibility that the winner's ex post IR fails is about a third.

$$\Pr(\text{winner's EPIR fails in } AGV) = \int_0^1 [\frac{1+\theta+\theta^2}{3} - \theta^2]d\theta = \frac{5}{18}$$

And the probability of winner being worse than loser is:

$$\Pr(\text{winner being worse than loser in } AGV)$$
$$= \int_0^{\frac{1}{2}\sqrt{3}-\frac{1}{2}} [\frac{2(1+\theta+\theta^2)}{3} - \theta^2]d\theta + \int_{\frac{1}{2}\sqrt{3}-\frac{1}{2}}^1 (1-\theta^2)d\theta = 0.60$$

If we look at ex post IC, we find for any realization of θ, there is at least one agent who wants to change the allocation and report.

Payoff Under M1.

If i winns, his ex post payoff is

$$u^w(\theta) = \frac{1}{2}\theta^2 - \frac{1}{8}\theta^2 = \frac{3}{8}\theta^2$$

if i loses, his ex post pay off is $u^w(\theta) = \frac{1}{8}\theta^2$. In both case, the participant's ex post payoff is positive. Meanwhile under $M1$, nobody would like to make change if an ex post realization falls into region $\theta_j \geq \sqrt{3}\theta_i$, which happens with probability $\frac{\sqrt{3}}{3} = 58\%$.

3.2 Asymmetric Distribution Case

We now consider environments where the distribution is not identical, but still independent. As we will see, the payment rule can be characterized by a set of non-homogeneous linear ordinary differential equations. Moreover, we can obtain some important observations without requirement to solve the system analytically. As a special case, when $n = 2$, we can completely solve the payment function without putting restriction on distributions.

With a little loss of generality, we assume that all θ_i's are in the same support $[\underline{\theta}, \overline{\theta}]$, being drawn according to c.d.f. $F_i(\theta_i)$ association with continuous differentiable p.d.f. $f_i(\theta)$.

Let $G_j(z_j) = \Pi_{k\neq j} F_k(z_j)$ be the c.d.f. of the random variable $z_j = \max_{k\neq j} z_k$, association with p.d.f. $g_j(z_j) = G'_j(z_j)$, and let $\Pi(z) = \Pi_{k=1}^n F_k(z)$. The distribution that j wins given i is not the winner is,

$$\begin{aligned} F_{n-1:n-1}(z_j + \Delta z \backslash z_j &> \theta_i) - F_{1:n-1}(z_j \backslash z_j > \theta_i) \\ &= \Pr(Z_j \in [z_j, z_j + \Delta z], z_j > \max_{k\neq i,j} z_k \backslash z_j > \theta_i) \\ &= \frac{\Pr(Z_j \in [z_j, z_j + \Delta z], z_j > \max_{k\neq i,j} z_k, z_j > \theta_i)}{\Pr(\max_{k\neq i} z_k > \theta_i)} \\ &= \frac{(F_j(z_j + \Delta z) - F_j(z_j)) \left[\Pi_{k\neq j,i} F_k(z_j)\right]}{1 - G_i(\theta_i)} \end{aligned}$$

Therefore,

$$f_{n-1:n-1}(z_j \backslash z_j > \theta_i) = \frac{f_j(z_j) \left[\Pi_{k\neq j,i} F_k(z_j)\right]}{1 - G_i(\theta_i)}$$

Let $M_i(\theta_i)$ be the winner i's payment when he reports θ_i, then the expected payment for i is:

$$m_i(\theta_i) = G_i(\theta_i) M_i(\theta_i) - \frac{1}{n-1} \sum_{j\neq i} \int_{\theta_i}^{\bar{\theta}} M_j(z) \left[\Pi_{k\neq j,i} F_k(z_j)\right] dF_j(z) \tag{5}$$

Note that $m_i(\theta_i)$ is incentive compatible if and only if

$$m_i'(\theta) = g_i(\theta)S_i(\theta)$$

therefore, we can characterize the payment rule $M_i(\theta_i)$ through the following ODEs. Take derivative w.r.t.θ_i on both sides of equation (5), obtain an ODE for any i,

$$S_i(\theta_i)\sum_{k\neq i}\frac{G_k f_k}{F_i} = M_i\sum_{k\neq i}\frac{G_k f_k}{F_i} + G_i M_i' + \frac{1}{n-1}\sum_{k\neq i}\frac{G_k f_k}{F_i}M_k$$

This is

$$S_i(\theta_i)\sum_{k\neq i}\frac{f_k}{F_k} = M_i\sum_{k\neq i}\frac{f_k}{F_k} + M_i' + \frac{1}{n-1}\sum_{k\neq i}\frac{f_k}{F_k}M_k$$

where we compress the augument θ for convenience.

Write $\frac{f_k}{F_k} = q_k$, $q_{-i} = \sum_{k\neq i} q_k$, denote $\mathbf{M}(\theta) = \begin{pmatrix} M_1(\theta) \\ M_2(\theta) \\ ... \\ M_n(\theta) \end{pmatrix}$,

$$\mathbf{A}(\theta) = -\begin{pmatrix} q_{-1}(\theta) & \frac{1}{n-1}q_2(\theta) & ... & \frac{1}{n-1}q_n(\theta) \\ \frac{1}{n-1}q_1(\theta) & q_{-2}(\theta) & \frac{1}{n-1}q_3(\theta) & \frac{1}{n-1}q_n(\theta) \\ ... & ... & ... & ... \\ \frac{1}{n-1}q_1(\theta) & \frac{1}{n-1}q_2(\theta) & \frac{1}{n-1}q_3(\theta) & q_{-n}(\theta) \end{pmatrix}$$

$\mathbf{B}(\theta) = \begin{pmatrix} q_{-1}(\theta) \\ q_{-2}(\theta) \\ ... \\ q_{-n}(\theta) \end{pmatrix} S(\theta)$. Finally, the ODE system is,

$$\mathbf{M}'(\theta) = \mathbf{A}(\theta)\mathbf{M}(\theta) + \mathbf{B}(\theta) \quad \text{with} \quad \mathbf{M}(\underline{\theta}) = \frac{1}{n}\mathbf{S}(\underline{\theta}) \tag{6}$$

The solution of the above non-homogenous linear ODE system with initial condition $M_i(\underline{\theta}) = \frac{1}{n}S(\underline{\theta})$ exists and is unique.

It seems hard to solve the above system in general if n is large. But for us, the information from the above characterization is enough for us to have the following important result.

Proposition 4 *Under asymmetric but independent environment, (i) the payment rule $M(\theta)$ characterized by (6) is ex post socially efficient association with allocation x^*; (ii) the lowest type's* *expected payoff coincides with expected surplus of VCG* $\sum U_i(\underline{\theta}) = E\Delta$. ***(Proof see Appendix A4)***.

The above proposition generalizes the result of SIPV. And the second property shows that under this payment rule the lowest type agents' social benefit is exactly the expected social surlus in VCG. (AGV does not have property (ii)).

Remark 3 *This observation can be generalized to the situation that utility function is also heterogenous. In that situation, we can re-parameterize the type and the similar reasoning still applies (with a little loss of generality in the support restriction).*

For intuition, we solve the explicit solution in a two player case, for any distribution. We have the following proposition.

Proposition 5 *Under environment of asymmetric independent private value (AIPV), the payment rule of M1 is,*

$$M_i(\theta_i) = \frac{1}{2}S(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \frac{f_j(\theta)\int_{\underline{\theta}}^{\theta} F_i(z)F_j(z)S'(z)dz}{F_i(\theta)F_j(\theta)^2} d\theta \tag{7}$$

and satisfies ex post monotonicity. ***(Proof see Appendix A5)***

Therefore, we find even under asymmetric distribution, the ex ante side-payment is not required to meet both interim IR and ex post IR. For intuition, we provide the following example.

Example 2 ***Asymmetric payment function when type distribution is a power function.***

If the distribution is $F_i = \theta^{a_i}$, *it can be calculated that,*

$$U_i(\underline{\theta}) = \frac{1}{\gamma}\left(\frac{a_j}{\gamma + a_j} - \frac{a_j}{a_j + a_i + \gamma}\right) > 0$$

So definitely, the above regime is ex post socially efficient although AGV is not interim soically efficient. We find that $M_i(\theta) \geq M_j(\theta)$ *iff* $a_j \geq a_i$.

$$M_i(\theta_i) = \int_{\underline{\theta}}^{\theta_i} \frac{a_j\theta^{a_j-1}\int_{\underline{\theta}}^{\theta} z^{a_i+a_j}z^{\gamma-1}dz}{\theta^{a_i+2a_j}} d\theta = \frac{a_j}{a_i + a_j + \gamma}\theta_i^{\gamma}$$

The stronger type agent's payment scheme will be flatter than the weaker one's, as we find in standard first price auction (Maskin and Riley, 2001).

As an important result, we now improve lemma 1 to a stronger version, by incorporating ex post IR.

Theorem 4 *If type is independent, under private value case, (i) there exist ex post socially efficient mechanisms if and only if VCG runs expected surplus, $E\Delta \geq 0$; (ii) in addition, M1 is the unique payment rule generates ex post monotonicity.*

Proof. (i) The necessary part is commonly known in the existing literature, same as the proof of lemma 1. We only show the sufficiency. First consider the payment structure $\mathbf{M}^F$, defined in **M1**. By revenue equivalence theorem, any two incentive compatible mechanims differ in their payoff up to a constant. So there exist constants c_i^F such that,

$$U_i^F(\theta_i) = \mathbb{E}_{\theta_{-i}} S(\theta_i, \boldsymbol{\theta}_{-i}) - c_i^F$$

It is also the case that constants c_i^V exist, such that,

$$U_i^V(\theta_i) = \mathbb{E}_{\theta_{-i}} S(\theta_i, \boldsymbol{\theta}_{-i}) - c_i^V$$

If VCG runs an expected surplus, $\mathbb{E}\Delta \geq 0$ means,

$$\mathbb{E} \sum c_i^V \geq \mathbb{E} \sum c_i^F$$

for all $i \geq 1$, define $d_i = c_i^F - c_i^V$, and let $d_1 = -\sum_{i=2}^n d_i$. Then we can construct a mechanism $\mathbf{M}^\#$ by

$$M_i^\#(\theta) = M_i^F(\theta) + d_i$$

and this means $\mathbf{M}^\#$ is also incentive compatible. We only need to check $\mathbf{M}^\#$ is ex post individual rational. Importantly, note that $M_i^F(\theta)$ is ex post individual rational, therefore, the difference of ex post payoff between $M_i^\#(\theta)$ and $M_i^F(\theta)$ is also up to a constant[7]. So it is sufficient to check the ex post using similar construction.

For $i \neq 1$,

$$U_i^\#(\theta_i) = U_i^F(\theta_i) + d_i = U_i^F(\theta_i) + c_i^F - c_i^V = U_i^V(\theta_i) \geq 0$$

and

$$\begin{aligned} U_1^\#(\theta_1) &= U_1^F(\theta_1) + d_1 \\ &\geq U_1^F(\theta_1) + d_1 \\ &= U_1^V(\theta_1) \\ &\geq 0 \end{aligned}$$

[7] This is the key part of the proof. In the proof of lemma 1 (Krishna and Perry, 1998; Krishna, 2002), their proof is based on AGV, because AGV *pe se* may not be IR ex post, $M_i^A(\theta) + d_i$ may not be IR ex post as well, although $M_i^A(\theta) + d_i$ may be IR interim.

Since both VCG and $< \mathbf{x}^*(\theta), M^F(\theta) >$ are ex post individual rational, then $\mathbf{M}^{\#}$ is also ex post individually rational.

(ii) The proof is similar to theorem 3. The key observation is that the payment must depend on the winner's type only. Q.E.D. ■

In fact, **M1** does not require ex ante side-payment, even under the asymmetric environment, which is an advantage in terms of pragmatic implimentation.

4 General Preference with Presence of Externality

Now we consider environment with presence of externality, either due to utility interaction like public good, or cost complementarity/substitution like spill-over. For tractability, here we discover a two-agent case, and will discuss an n-agent generalization later. It will be shown that the mechanism works in a 2-agent case might not work in the n-agent case. Interestingly, there is dramatically different implication between endogenous endowment situtwithout and its exogeneous countpart[8]. So we deal with them separately.

4.1 Endogenuous Quantity

4.1.1 Basics

For notational convenience, let

$$f^{(n-1)}(\theta) = (n-1)! f(\theta_{n-1:n-1}) ... f(\theta_{1:n-1})$$

be the joint pdf of n-1 order statatistics. And we deonte the expected consumption to i, when i's type is ranked as j-th order statistic as:

$$\bar{v}_i^{(j)}(x_i^*(.), \theta) = \underbrace{\int_{\theta}^{\bar{\theta}} ... \int_{\theta}^{\theta_{j-2}}}_{j-1} \underbrace{\int_{\underline{\theta}}^{\theta} \{ \int_{\underline{\theta}}^{\theta_{j+1}} \int_{\underline{\theta}}^{\theta_{j+2}} ... \int_{\underline{\theta}}^{\theta_{n-1}}}_{n-j} v_i(x_i^*(.., \theta_{j-1}, \theta, \theta_{j+1}, ..), \theta) f^{(n-1)}(\theta_{-i}) d\theta_{-i}.$$

Similarly, his expected social surplus is,

$$\bar{S}^{(j)}(\theta) = \underbrace{\int_{\theta}^{\bar{\theta}} ... \int_{\theta}^{\theta_{j-2}}}_{j-1} \underbrace{\int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_{j+1}} \int_{\underline{\theta}}^{\theta_{j+2}} ... \int_{\underline{\theta}}^{\theta_{n-1}}}_{n-j} S(.., \theta_{j-1}, \theta, \theta_{j+1}, ..) f^{(n-1)}(\theta_{-i}) d\theta_{-i}$$

[8] When endowment is endogenous, $S(\boldsymbol{\theta})$ will be supermodular, while endowment is exogeneous, $S(\boldsymbol{\theta})$ will be submodular, see proposition 1.

We propose the mechanism as follows:

M2: *(i) The designer chooses optimal alloction rule* $\mathbf{x}^*(\hat{\boldsymbol{\theta}})$ *according to report* $\hat{\boldsymbol{\theta}}$*;*

(ii) each player receives consumption $x_i^*(\hat{\theta}_i, \hat{\theta}_{-i})$*, and the payment rule is the following:*

$$M_i^k(\hat{\theta}_i, \hat{\theta}_{-i}) = \begin{cases} -S_{-i}(\hat{\theta}_i, \hat{\theta}_{-i}) + [(1-k)\beta^k(\hat{\theta}_i) + k\beta^k(\hat{\theta}^{n-2:n})] & \text{if } \hat{\theta}_i > \max_{j \neq i} \hat{\theta}_j \\ \frac{1}{\# \mathcal{N}_j} \sum [S(\hat{\theta}_i, \hat{\theta}_{-i}) - \beta^k(\hat{\theta}_i)] & \text{if } \hat{\theta}_i = \hat{\theta}_j \text{ for } i, j \in \mathcal{N}_j \\ v_i(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_i) - \frac{[(1-k)\beta^k(\hat{\theta}^{n:n}) + k\beta^k(\hat{\theta}^{n-1:n})]}{n-1} & \text{if } \hat{\theta}_i < \max_{j \neq i} \hat{\theta}_j \end{cases}$$

where

$$\beta^k(\theta) = \frac{\int_{F^{-1}(k)}^{\theta} \left(\frac{d}{d\tau} \bar{S}^{(1)}(\tau) - \sum_{j=1}^{n} \frac{d}{d\tau} \bar{v}_i^{(j)}(x_i^*(.,\tau,.), \tau) + m'(\tau) \right) \frac{(F(\tau)-k)^{n-1}}{F(\tau)^{n-2}} d\tau}{(F(\theta) - k)^n} \tag{8}$$

with $m'(\tau) = \left[\frac{\partial}{\partial z} \int_{\theta_{-i}} \frac{\partial}{\partial x} v(x(z, \theta_{-i}), z) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \right]_{z=\tau}$ and with $k \in [0,1]$ as a constant.

Under this mechanism, the individual with type θ_i pretends to report $\tilde{\theta}_i$ will have the following ex post payoff:

$$u_i(\theta_i, \tilde{\theta}_i) = \begin{cases} v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) + S_{-i}(\tilde{\theta}_i, \theta_{-i}) - [(1-k)\beta^k(\theta_i) + k\beta^k(\theta^{n-1:n})] & \text{if } \theta_i > \max_{j \neq i} \theta_j \\ \frac{1}{\# \mathcal{N}_j} \sum [S(\theta_i, \theta_{-i}) - \beta_i] & \text{if } \theta_i = \theta_j \text{ for } i, j \in \mathcal{N}_j \\ v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - v_i(x_i(\tilde{\theta}_i, \theta_{-i}), \tilde{\theta}_i) + \frac{[(1-k)\beta^k(\theta^{n:n}) + k\beta^k(\theta^{n-1:n})]}{n-1} & \text{if } \theta_i < \max_{j \neq i} \theta_j \end{cases}$$

The above mechanism can be understood as follows. The highest type agent gets the entitlement to charge all social surplus at the cost of paying lump sum payment to the remaining losers; while the remaining losers need to pay the consumptions (thus earn zero consumption surplus) according to their reports but are paid by lump sum transfer from the winner. We claim that (i) the mechanism **M2** is interim socially efficient for any $k \in [0,1]$ if and only if $\mathbb{E}\Delta \geq 0$, and (ii) there exist some k such that **M2** is ex post socially efficient.

The following theorem states the results regarding (i).

Theorem 5 *If the distribition is i.i.d., and utility is symmetric, mechanism M2:* $< x^*(\theta), M^k(\theta) >$ *is: (i) budget balance and incentive compatible, and (ii) interim socially efficient if and only if* $\mathbb{E}\Delta \geq 0$ *for any* $k \in [0,1]$*.* ***(Proof see Appendix A5)***

Under **M2**, the lowest type agent's payoff is exactly a $\frac{1}{n}$ share of total VCG expected social surplus (deficit). Obviously, **M2** is one kind of budget balance mechanism other than AGV. Compared with AGV, **M2** is interim budget balance if and only if VCG runs expected social surplus[9].

Remark 4 *$\beta^k(\theta)$ may be negative or even non-monotonic. If $\beta^k(\theta)$ is negative, this means the remaining people need to subsidize the winner in equilibrium.*

4.1.2 Two-player case

If $n = 2$, we claim that $k = 1$ is expost socially efficient. It is easy to know that from formula (8),

$$\beta^1(\theta) = \frac{\int_\theta^{\bar\theta}(1 - F(\tau))(S(\tau, \tau) - h(\tau))dF(\tau)}{[1 - F(\theta)]^2}$$

where

$$h(\tau) \equiv \frac{\int_\tau^{\bar\theta} \frac{\partial}{\partial\tau} v(x^*(\tau, z), \tau)dF(z)}{f(\tau)}$$

The following theorem states the result regarding **M2** when $n = 2$.

Theorem 6 *Given that endowment is endogenous, if distribution is i.i.d. and preference is symmetric, when $n = 2$, M2 with $k = 1$ has the following properties: (i) lump sum payment function $\beta^1(\theta)$ is monotonic (without restriction on distribution); (ii) M2 is ex post socially efficient if and only if $\mathbb{E}\Delta \geq 0$. and (iii) M2 also satisfies ex post monotonicity*

Proof. (i) If the endownment is endogenous, then $\frac{\partial}{\partial\tau} v(x^*(\tau, z), \tau)$ is an increasing function of z, since $x^*(\tau, z)$ is an increasing function of z. Therefore,

$$\begin{aligned}
\beta^1(\theta) &= \frac{\int_\theta^{\bar\theta}(1 - F(\tau))(S(\tau, \tau) - h(\tau))dF(\tau)}{[1 - F(\theta)]^2} \\
&\leq \frac{\int_\theta^{\bar\theta}(1 - F(\tau))(S(\tau, \tau) - \frac{\partial}{\partial\tau} v(x^*(\tau, \tau), \tau)\frac{\int_\tau^{\bar\theta} dF(z)}{f(\tau)})dF(\tau)}{[1 - F(\theta)]^2} \\
&= \frac{\int_\theta^{\bar\theta}(1 - F(\tau))S(\tau, \tau)dF(\tau)}{[1 - F(\theta)]^2} - \frac{\frac{1}{2}\int_\theta^{\bar\theta} \frac{dS(\tau,\tau)}{d\tau}(1 - F(\tau))^2 d\tau}{[1 - F(\theta)]^2} \\
&= \frac{1}{2}S(\theta, \theta)
\end{aligned}$$

[9] To see this, note in symmetric situation, under AGV, the lowest type's payoff is,

$$\begin{aligned}
U^A(\underline{\theta}_i) &= \mathbb{E}_{\theta_{-i}} S(\underline{\theta}_i, \theta_{-i}) - \frac{1}{n-1}\mathbb{E}_{\theta_{-i}} \sum_{j \neq i} \mathbb{E}_{\theta_{-j}}[S_{-j}(\theta_j, \theta_{-j})] \\
&= \mathbb{E}_{\theta_{-i}} S(\underline{\theta}_i, \theta_{-i}) - \frac{(n-1)}{n}[\mathbb{E}_{\boldsymbol\theta} S(\theta_i, \theta_{-i}) + \mathbb{E}_{\theta_{-i}}\mathbb{E}_{\theta_{-j}} c(\mathbf{x}(\theta_j, \theta_{-j}))] \\
&\leq \mathbb{E}_{\theta_{-i}} S(\underline{\theta}_i, \theta_{-i}) - \frac{(n-1)}{n}\mathbb{E}_{\boldsymbol\theta} S(\theta_i, \theta_{-i})
\end{aligned}$$

The last step comes from integration by parts. Note that when $\theta \to \overline{\theta}$,

$$\beta^1(\bar{\theta}) = \frac{1}{2} S(\bar{\theta}, \bar{\theta})$$

This means $\beta^1(\theta)$ is uniformly bounded by $\frac{1}{2}S(\theta, \theta)$, but achieves $\frac{1}{2}S(\theta, \theta)$ at the boundary, which implies that

$$\beta'(\theta) \geq \frac{1}{2}\frac{d}{d\theta} S(\theta, \theta) > 0$$

Suppose this is not true, then for some $\theta < \bar{\theta}$, there always exists ϵ small enough such that for $\theta \in [\bar{\theta} - \epsilon, \bar{\theta}]$, $2\beta(\theta) > S(\theta, \theta)$, a contradiction.

(ii) We only need to check ex post IR constraint. To meet the loser's ex post IR, the bid should be non-negative. To see this,

$$\begin{aligned}
& \beta^1(\underline{\theta}) \\
= & \int_{\underline{\theta}}^{\bar{\theta}} [S(\theta, \theta) - h(\theta)](1 - F(\theta)) dF(\theta) \\
= & \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} S(\theta, \tau) dF(\tau) dF(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \frac{1 - F(\theta)}{f(\theta)} \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} v(x^*(\theta, \tau), \theta) dF(\tau) dF(\theta) \\
& + \int_{\underline{\theta}}^{\bar{\theta}} \frac{1 - F(\theta)}{f(\theta)} \int_{\theta}^{\bar{\theta}} \frac{\partial}{\partial \theta} v(x^*(\theta, \tau), \theta) dF(\tau) dF(\theta) \\
= & \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} S(\theta, \tau) dF(\tau) dF(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{\lambda(\theta)} \frac{\partial}{\partial \theta} v(x^*(\theta, \tau), \theta) dF(\tau) dF(\theta) \\
= & \frac{1}{2} \mathbb{E}\Delta = U(\underline{\theta})
\end{aligned}$$

As long as $\mathbb{E}\Delta \geq 0$, then the bid will be non-negative overall, and the lowest type agent's payoff will be non-negative overall (not only interim but also ex post). For the winner, it is easy to see for any $\theta_i > \theta_j$,

$$S(\theta_i, \theta_j) - \beta(\theta_j) > 0$$

since $\beta(\theta_j) \leq \frac{1}{2}S(\theta_j, \theta_j)$.

(iii) It is ready to see, from the monotonicity of the bid, that the loser's ex post pay-off is monotonic over his own type, and the winner's pay-off is monotonic over his own type too since payment is independent of his own type. The only point we need to check is the pivotal point. As $\theta_j \to \theta_i$, the winner still be better off since $\beta(\theta_j) \leq \frac{1}{2}S(\theta_j, \theta_j)$. The equality only holds at $\theta = \overline{\theta}$. Q.E.D. ■

The above theorem says that for a two-person case, mechanism **M2** solves the allocation problem well. And we verify that the necessary and sufficient condition for existence of an ex post socially efficient mechanism is the same as that of interim socially efficient one. It is worth pointing out that $\beta^0(\theta)$ might not be always ex post individually rational, due to the fact

$$S(\theta, \underline{\theta}) - \beta(\theta) \geq 0$$

might fail when $\beta(\underline{\theta}) = \mathbb{E}\Delta > 0$ but $S(\underline{\theta}, \underline{\theta}) = 0$.

Remark 5 *If the outside reservation utility $\underline{u}_i(\theta_i)$ is symmetric but type-dependent, then we need to modify the social welfare $S(\theta)$ as net social welfare $S(\theta) - \sum \underline{u}_i(\theta_i)$, and each agent's gain from the project will be net gain $v_i(x_i^*, \theta_i) - \underline{u}_i(\theta_i)$. The conclusions from the above theorem are still true as we will see in the next section.*

Do other values of $k \in [0, 1)$ have such kind of properties? For example, $k = 0$, The following corollary states the result.

Corollary 1 *For M2 when $n = 2$, under the same condition as theorem 4, there exists a cut-off $k^* \in [0, 1)$ such that for $k \geq k^*$, M2 with $k < 1$ possesses the same properties as $k = 1$ if $\mathbb{E}\Delta > 0$.*

Proof. The proof crucially depends on the property of $k = 1$. Note that $\beta^{k\prime}(\theta)$ is a continous function of k and $\beta^{k\prime}(\theta) > 0$ for $k = 1$, therfore, for k close enough to 1, $\beta^{k\prime}(\theta) \geq 0$ will still hold. At the same time, observing that

$$\begin{aligned} h'(\theta) &= \frac{-\frac{\partial v}{\partial \theta} v(x^*(\theta,\theta),\theta) f(\theta) + \int_\theta^{\bar{\theta}} \frac{d}{d\theta}\left(\frac{\partial v}{\partial \theta} v(x^*(\theta,\tau),\theta)\right) dF(\tau)}{f(\theta)} \\ &\quad - \frac{f'(\theta)}{f(\theta)} \frac{\int_\theta^{\bar{\theta}} \frac{\partial v}{\partial \theta} v(x^*(\theta,\tau),\theta) dF(\tau)}{f(\theta)} \end{aligned}$$

will be negative as θ close enough to $\bar{\theta}$, which means for k close enough to 1, for $\theta \geq F^{-1}(k)$, we have

$$\begin{aligned} \beta^{k\prime}(\theta) &= \frac{f(\theta)}{F(\theta) - k}[S(\theta,\theta) - h(\theta) - 2\frac{\int_{F^{-1}(k)}^\theta (F(\tau) - k)(S(\tau,\tau) - h(\tau)) dF(\tau)}{[F(\theta) - k]^2}] \\ &= \frac{f(\theta)}{F(\theta) - k} \frac{\int_{F^{-1}(k)}^\theta (F(\tau) - k)^2 (\frac{d}{d\tau} S(\tau,\tau) - h'(\tau)) d\tau}{[F(\theta) - k]^2} \geq 0 \end{aligned}$$

Moreover, if $\mathbb{E}\Delta > 0$, therefore, it is possible to have $\beta(\underline{\theta}) \geq 0$ for k to be close enough to 1. In sum, take $k^* = \sup_k\{k : \beta'(\theta) \geq 0 \text{ and } \beta(\underline{\theta}) \geq 0\}$, therefore, $\beta^k(\theta)$ will be monotonic and non-negative.

It is also easy to see that $\beta^k(\theta) \leq \frac{1}{2}S(\theta,\theta)$ by the similar derivation:

$$\begin{aligned}
\beta^k(\theta) &= \frac{\int_{F^{-1}(k)}^{\theta}(F(\tau)-k)(S(\tau,\tau)-h(\tau))dF(\tau)}{[F(\theta)-k]^2} \\
&\leq \frac{\int_{F^{-1}(k)}^{\theta}(F(\tau)-k)(S(\tau,\tau)-\frac{\partial}{\partial\tau}v(x^*(\tau,\tau),\tau)\frac{\int_{\tau}^{\bar{\theta}}dF(z)}{f(\tau)})dF(\tau)}{[F(\theta)-k]^2} \\
&= \frac{\int_{F^{-1}(k)}^{\theta}(F(\tau)-k)\left[S(\tau,\tau)-\frac{\partial}{\partial\tau}v(x^*(\tau,\tau),\tau)\frac{1-F(\tau)}{f(\tau)}\right]dF(\tau)}{[F(\theta)-k]^2} \\
&= \frac{1}{2}S(\theta,\theta) - \frac{\frac{1}{2}(1-k)\int_{F^{-1}(k)}^{\theta}\frac{dS(\tau,\tau)}{d\tau}(F(\tau)-k)d\tau}{[F(\theta)-k]^2} \\
&\leq \frac{1}{2}S(\theta,\theta)
\end{aligned}$$

Q.E.D. ■

For intuition, we provide a concrete example.

Example 3 *Two residents are living in a small town. They can build a public good (like internet) together and share with each other or build the good on their own. Suppose that the utility function for each individual i over the size of public good x is* $v(x,\theta_i) = \theta_i(2x - x^2)$, *and the cost of x is* $c(x) = cx^2$. *We assume type* θ_i *drawn from* $[a, 1]$ *with c.d.f.* $F(\theta) = \frac{\theta-a}{1-a}$. *If they build the good autarkily, they choose* $x_i(\theta_i) \in \arg\max_x v(x,\theta_i)$, *particularly,* $\underline{u}(\underline{\theta}) = \max_x v(x,\underline{\theta})$[10]. *Does there exist an ex post socially efficient mechanism for them to cooperate? Note that*

$$x^*(\boldsymbol{\theta}) = \frac{\sum\theta_i}{c+\sum\theta_i}$$

and

$$S(\boldsymbol{\theta}) = \frac{(\sum\theta_i)^2}{c+\sum\theta_i}$$

In this example, use $c = 1$, $a = \frac{1}{2}$, *and it can be computed* $\mathbb{E}\Delta = 4(6\ln 5 - 9\ln 2 - 3\ln 3) = 0.490 > 2u(\underline{\theta}) = \frac{1}{3}$.

For M2 with k=1, we have

$$h(\tau) \equiv \frac{\int_{\tau}^{1}\frac{\partial}{\partial\tau}v(x^*(\tau,z),\tau)dF(z)}{f(\tau)} = 1-\tau-\frac{1}{1+2\tau}+\frac{1}{2+\tau}$$

[10]We can verify that $\arg\min_{\theta_i}\mathbb{E}S(\theta_i,\theta_j) - \underline{u}(\theta_i) = \underline{\theta} = \frac{1}{2}$ due to monotonicity of $\mathbb{E}S(\theta_i,\theta_j) - \underline{u}(\theta_i)$ in θ_i.

and

$$\beta^1(\theta) = \frac{\int_\theta^1 (1-F(\tau))(S(\tau,\tau)-h(\tau))dF(\tau)}{[1-F(\theta)]^2}$$
$$= \frac{\int_\theta^1 (1-\tau)(\frac{4\tau^2}{1+2\tau} - (1-\tau-\frac{1}{1+2\tau}+\frac{1}{2+\tau}))d\tau}{(1-\theta)^2}$$
$$= -\frac{1}{2(1-\theta)^2}[1+[\theta(5-2\theta)-4]\theta + \ln 27 - 6\ln(\theta+2) + 3\ln(1+2\theta)]$$

It can be shown that $\beta^1(\theta)$ is monotone and non-negative (In the following Fig. The red dot line is $\frac{1}{2}S(\theta,\theta)$, the green dots line is $\underline{u}(\theta)$ and the black solid line is $\beta^1(\theta)$).

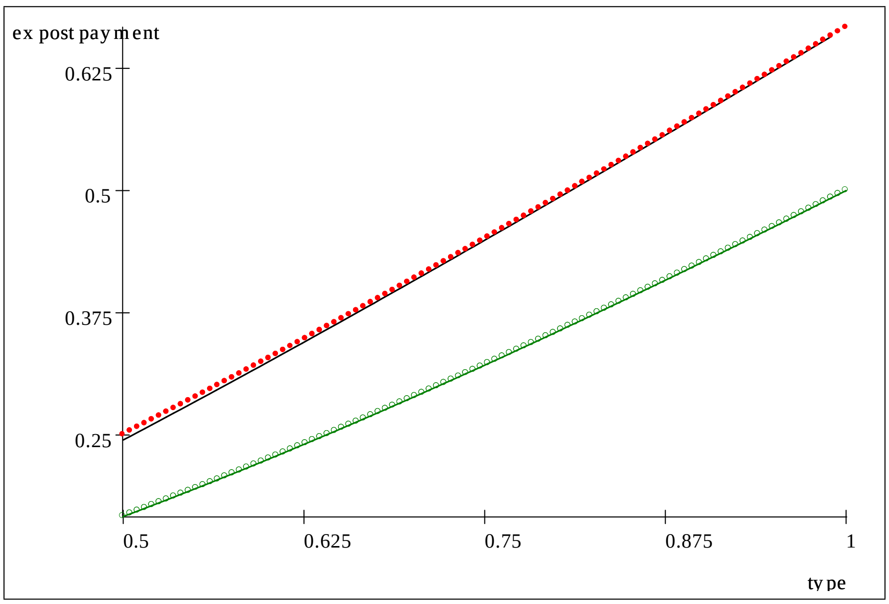


Fig. 2. Ex post payoff functions

To verify ex post IR, we find $\beta^1(\theta) - \underline{u}(\theta) > 0$, and $s(\theta_i,\theta_j) - \beta^1(\theta_j) > \underline{u}(\theta_i)$ for any $\theta_i > \theta_j$ by noting that

$$\frac{(\sum \theta_i)^2}{1+\sum \theta_i} - \frac{\theta_i^2}{1+\theta_i} > \frac{2\theta_j^2}{1+2\theta_j}$$

since the LHS of the above inequality is an increasing function of θ_i.

We also take a look at M2 with $k=0$. The payment function is,

$$\beta^0(\theta) = \frac{\int_{\frac{1}{2}}^{\theta} F(\tau)(S(\tau,\tau)-h(\tau))dF(\tau)}{F(\theta)^2}$$
$$= \frac{2(-\frac{3}{8}+2\theta-\frac{7}{2}\theta^2+2\theta^3-\ln\frac{3125}{128}+5\ln(2+\theta)-2\ln(1+2\theta))}{(1-2\theta)^2}$$

Therefore, ex post IR

$$\underline{u}(\theta) \le \beta^0(\theta) \le S(\theta,\underline{\theta}) - \underline{u}(\theta)$$

fails.

Meanwhile, M3 with $k = 0$ proposed by the next subsection is also not ex post individual rational i.e., the payment or bid does not satisfy

$$\underline{u}(\theta) \leq b^0(\theta) \leq S(\theta, \theta) - \underline{u}(\theta)$$

Although when $\underline{u}(\theta) = 0$, it will be true, $0 \leq b^0(\theta) \leq S(\theta, \theta)$.

Remark 6 *The mechanism considered by CGK is not ex post individually rational, even proceeded with a side-payment. One of the advantages of M2 is without need of side-payments.*

4.2 Fixed Endowment

4.2.1 Basics

When the endowment is fixed, then VCG always runs expected social surplus. This observation comes from the following:

$$\begin{aligned}
\mathbb{E}\Delta &= \int_{\boldsymbol{\theta}} S(\boldsymbol{\theta}) d\mathbf{F}(\boldsymbol{\theta}) - \sum \int_{\theta_{-i}} \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} (1 - F_i(\theta_i)) \frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i \right) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \\
&= \sum \left(\int_{\boldsymbol{\theta}} v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) d\mathbf{F}(\boldsymbol{\theta}) - \int_{\theta_{-i}} \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} (1 - F_i(\theta_i)) \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i} d\theta_i \right) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \right) \\
&\geq \sum \left(\int_{\boldsymbol{\theta}} v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) d\mathbf{F}(\boldsymbol{\theta}) - \int_{\theta_{-i}} \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} (1 - F_i(\theta_i)) dv_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) \right) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \right) \\
&= \sum \int_{\theta_{-i}} v_i(x_i^*(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \\
&\geq 0
\end{aligned}$$

The reason that $\frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} = \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}$ is similar to lemma 3; the third equality is due to $\frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i} \leq \left[\frac{\partial v_i(x_i^*(z_i, \theta_{-i}), \theta_i)}{\partial z_i} \right]_{\theta_i = z_i} + \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}$.

However, **M2** is no longer ex post individually rational. To see this, take $n = 2$ as an example, it is noted that now $\frac{\partial}{\partial z} v(x^*(\tau, z), \tau)$ is a decreasing function of z. It is no longer the case that theorem 5 holds in general[11]. We propose the following allocation rule (Reverse Order Allocation).

M3: *(i) The designer chooses the optimal allocation rule $\mathbf{x}^*(\hat{\boldsymbol{\theta}})$ according to report $\hat{\boldsymbol{\theta}}$;*

[11] For example, suppose $v(x_i, \theta_i) = \theta_i \sqrt{x_i}$, $\theta_i \tilde{} U[0, 1]$, $\sum x_i = 1$, then $S(\theta_i, \theta_j) = \sqrt{\sum \theta_i^2}$, in this case, it is impossible to have either $\beta^1(\theta) < S(\theta, \theta)$ or $\beta^0(\theta) > 0$ overall.

(ii) each player receives consumption $x_i^*(\hat{\theta}_i, \hat{\theta}_{-i})$, and the payment rule is the following:

$$M_i^r(\hat{\theta}_i, \hat{\theta}_{-i}) = \begin{cases} S(\hat{\theta}_i, \hat{\theta}_{-i}) - [(1-k)r^k(\theta_i) + kr^k(\hat{\theta}^{2:n})] & \text{if } \hat{\theta}_i < \min_{j \neq i} \hat{\theta}_j \\ \frac{1}{\# \mathcal{N}_j} \sum \{S(\hat{\theta}_i, \hat{\theta}_{-i}) - [(1-k)r^k(\theta_i) + kr^k(\hat{\theta}^{2:n})]\} & \text{if } \hat{\theta}_i = \min_{j \neq i} \hat{\theta}_j \text{ for } i, j \in \mathcal{N}_j \\ -v_i(x_i^*(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_i) + [(1-k)r^k(\hat{\theta}^{1:n}) + kr^k(\hat{\theta}^{2:n})] & \text{if } \hat{\theta}_i > \min_{j \neq i} \hat{\theta}_j \end{cases}$$

where

$$r^k(\theta) = \frac{\int_{F^{-1}(1-k)}^{\theta} \frac{[k-1+F(\tau)]^{n-1}}{(1-F(\tau))^{n-2}} (\sum_{j=1}^{n} \frac{d}{d\tau} \bar{v}_i^{(j)}(x_i^*(\tau, .), \tau) - m'(\tau) - \frac{d}{d\tau} \bar{S}^{(n)}(\tau)) d\tau}{[k-1+F(\theta)]^n} \tag{9}$$

Similarly, it can be justified that the above mechanism is (i) budget balanced and incentive compatible, and (ii) interim socially efficient if and only if $\mathbb{E}\Delta \geq 0$ for any $k \in [0, 1]$. **(See Appendix A6 for detail)**.

4.2.2 Two-player case

We claim M3 with $k = 1$ has nice properties in dealing with an exogeneous endowment case.

When $n = 2$,

$$r(\theta) = \frac{\int_{F^{-1}(1-k)}^{\theta} [k-1+F(\tau)](S(\tau, \tau) + \mu(\tau)) dF(\tau)}{[k-1+F(\theta)]^2}$$

where

$$\mu(\tau) = \frac{\int_{\underline{\theta}}^{\tau} \frac{\partial}{\partial \tau} v(x^*(\tau, z), \tau)) dF(z))}{f(\tau)}$$

And truth telling is a globally optimal strategy.

Theorem 7 *When $n = 2$, M3 is ex post socially efficient and satisfies ex post monotonicity.*

Proof. It is obvious that $r^1(\theta)$ is non-negative. And note that

$$\begin{aligned} r^1(\theta) &= \frac{\int_{\underline{\theta}}^{\theta} F(\tau)(S(\tau, \tau) + \mu(\tau)) dF(\tau)}{F(\theta)^2} \\ &\leq \frac{\int_{\underline{\theta}}^{\theta} F(\tau) S(\tau, \tau) dF(\tau) + \int_{\underline{\theta}}^{\theta} F(\tau)^2 \frac{d}{d\tau} S(\tau, \underline{\theta}) d\tau}{F(\theta)^2} \\ &= \frac{\int_{\underline{\theta}}^{\theta} F(\tau) S(\tau, \tau) dF(\tau) + F(\theta)^2 S(\theta, \underline{\theta}) - 2 \int_{\underline{\theta}}^{\theta} F(\tau) S(\tau, \underline{\theta}) dF(\tau)}{F(\theta)^2} \\ &\leq S(\theta, \underline{\theta}) \end{aligned}$$

The last step is due to submodularity $S(\underline{\theta}_i, \theta_j) \geq \frac{1}{2}[S(\theta_j, \theta_j) + S(\underline{\theta}_i, \underline{\theta}_i)] \geq \frac{1}{2} S(\theta_j, \theta_j)$[12]. Thus $S(\theta_i, \theta_j) - r^1(\theta_j) \geq 0$ for any $\theta_i \leq \theta_j$.

[12] This is due to the fact that $x_i^*(\theta_i, \theta_j), \theta_i)$ must decrease with θ_j since $\sum x_i = \bar{x}$. Therefore, based on supmodularity of $v(x_i^*(\theta_i, \theta_i), \theta_i)$, $S(\theta_i, \theta_j) + S(\theta_j, \theta_i) \geq S(\theta_i, \theta_i) + S(\theta_j, \theta_j)$.

To verify the ex post monotonicity, first of all, note that

$$\begin{aligned} r^1(\theta) &= \frac{\int_{\underline{\theta}}^{\theta} F(\tau)(S(\tau,\tau)+\mu(\tau))dF(\tau)}{F(\theta)^2} \\ &= \frac{1}{2}S(\theta,\theta) + \frac{2\int_{\underline{\theta}}^{\theta} F(\tau)\mu(\tau)dF(\tau) - \int_{\underline{\theta}}^{\theta} F(\tau)^2 \frac{d}{d\tau}S(\tau,\tau)d\tau}{2F(\theta)^2} \\ &\geq \frac{1}{2}S(\theta,\theta) + \frac{2\int_{\underline{\theta}}^{\theta} F(\tau)^2 \frac{\partial}{\partial\tau}v(x^*(\tau,\tau),\tau))d\tau - \int_{\underline{\theta}}^{\theta} F(\tau)^2 \frac{d}{d\tau}S(\tau,\tau)d\tau}{2F(\theta)^2} \\ &= \frac{1}{2}S(\theta,\theta) \end{aligned}$$

which implies $r^{1\prime}(\theta) \geq \frac{1}{2}\frac{d}{d\theta}S(\theta,\theta) \geq 0$ since $r^1(\underline{\theta}) = \frac{1}{2}S(\underline{\theta},\underline{\theta})$. Q.E.D. ■

However, $r^0(\theta)$ might not be always ex post individually rational. Note that the ex post individual rationality requires,

$$S(\theta_i,\theta_j) - r^0(\theta_i) \geq 0 \text{ for any } \theta_j \geq \theta_i.$$

When $\theta_j \to \theta_i \to \underline{\theta}$, it is possible that $S(\underline{\theta}_i,\underline{\theta}_j) - r^0(\underline{\theta}) = S(\underline{\theta}_i,\underline{\theta}_j) - \mathbb{E}\Delta < 0$ when $\mathbb{E}\Delta > 0$ but $S(\underline{\theta}_i,\underline{\theta}_j) = 0$.

Example 4 *The utility is the same as in example 3, but* $\Sigma x_i = 1$. *We have*

$$S(\theta) = \sum \theta_i - \frac{\theta_i\theta_j}{\theta_i+\theta_j}$$

$$x_i(\theta_i) = \frac{\theta_i}{\theta_i+\theta_j}$$

$$\mathbb{E}\Delta = 2\mathbb{E}S(0,\theta) - \int_0^1\int_0^1 \left(\theta_1+\theta_2-\frac{\theta_1\theta_2}{\theta_1+\theta_2}\right)d\theta_1 d\theta_2 = 2-0.795 > 0$$

And $\mu(\tau) = (\ln 4 - \frac{1}{2})\tau$ *and* $r^0(\theta) = \frac{\ln 4+1}{6}(2\theta+1)$. *So* $r^0(\theta)$ *is not ex post IR. But* $\frac{1}{2}S(\theta,\theta) = \frac{3}{4}\theta < r^1(\theta) = \frac{\ln 4+1}{3}\theta \leq S(\underline{\theta},\theta) = \theta$, *therefore* $r^1(\theta)$ *is ex post IR and satisfies ex post monotonicity. In fact, as we will show later, there is a window of distribution of endowment satisfying ex post IR even if the outside reservation is type-dependent.*

4.3 Discussion for case of n>2

The above allocation rules do not work well when $n > 2$. The issue is that $\beta^k(\theta)$ or $r^k(\theta)$ is no longer always bounded (either from below or above). Take $\beta^1(\theta)$ as an example, when $\theta \to \underline{\theta}$, $\beta^1(\underline{\theta})$ will be not bounded in general, due to the fact that item $\frac{\int_{\theta_{-i}} \frac{\partial}{\partial\tau}S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})}{F(\theta)^{n-2}}$ will dominate all other items when $n > 2$ as $\theta \to \underline{\theta}$. The economic intuition is that, due to the externality, the subsidy to the boundary type will be too high, once the number of player is more than 2. This situation happens not only in $\beta^k(\theta)$ or $r^k(\theta)$, but also under any allocation rule such as giving the entitlement to the j-th order highest type. Whether $\beta^k(\theta)$ or $r^k(\theta)$ is still valid depends on the functional form of utility $v(x,\theta)$. This fact demonstrates the sensitivity of the choice of a mechanism.

5 Auction-like Implementation and Bilateral Trade

5.1 Implement mechanism by auction

The mechanisms proposed in previous sections, M1, M2, or M3, can be implemented by a realisitic form of auction, especially under symmetric independent environment. For example, the first price auction with post auction redistribution will exactly implement the payment rule in M1 under SIPV. To see this, let b be the bidding function, then under the first price rule, the payoff is:

$$u_i(\theta_i) = \begin{cases} S(\theta_i) - b_i & \text{if } b_i > \max_{j\neq i} b_j \\ \frac{1}{\#\mathcal{N}_j}\sum[S(\theta_i) - b_i] & \text{if } \ b_i = b_j \ \forall i,j \in \mathcal{N}_j \\ \frac{1}{n-1}\max_{j\neq i} b_j & \text{if } \ b_i < \max_{j\neq i} b_j \end{cases} \tag{10}$$

In equilibrium, the optimal bidding strategy will be consistent with $\frac{n-1}{n}\mathbb{E}[S(\tau^{n:n})/\tau^{n:n} \leq \theta]$. However, the ex post individual rationality will be senstive to the payment rule. For example, the second price auction in our context will not be ex post individually rational, though it is still interim individually rational. In general, we can state our auction rule as follows:

(1) Government/designer runs a sealed price auction to pick a winner, and deterimines the supply of products according to the collected messages.

(2) The winner pays the cost, but gets the entitlement to charge the remaining bidders according to their consumption given their announced types;

(3) In order to obtain this entitlement, the winner(s) pays a lump sum transfer to the remaining losers. The rule of the game is known before bidding starts (in M2 who bids the highest wins; but in M3 who bid lowest wins).

(4) The revenue (deficit) collected from the winner's payment is redistributed among all bidders (perhaps including the winner himself), the redistribution policy is known before bidders submit their bids.

Our advice for chioce of auction rule can be summarized as the following.

Table 2: Chioce of auction rule in different context

	Private good	Public good	Public good
Endowment		Endogeneous	Exogeneous
Title Allocation	The highest type	The highest type	The lowest type
Price Rule	1st price auction	Second price auction	Second price auction
# of Bidder	$n \geq 2$	$n = 2$	$n = 2$

In SIPV environment, based on our first price auction, the auctioneer can earn a risk-free revenue that is the same as any efficient auction in terms of expectation. So if the auctioneer is risk averse, he can induce the bidders to compete for the entitlement, and only charge the participants an entry fee. The participants then will behave exactly as what we have described in section 3. And this auction will be ex post individually rational, so that nobody will quit due to an outside option. Another important implication is that this auction form is also collusion-proof, from the auctioneer's prespective.

Moreover, in public good case, the present auction form possesses several important advantages, compared with auction of share when the quantity is continuous. In continuous quantity or multi-unit auction case, the bidder needs to submit their demand function, and in general, the allocation is inefficient (Wilson, 1979, Ausubel, 2004). However, in our formulation, the allocation is efficient, and the auctioneer's revenue is maximized among all efficient allocations. For intuition, we provide the following comparisons.

Suppose the auctioneer has one unit good for sale, say, $\sum x_i = 1$. The bidder i's utility from x_i is $2\theta_i\sqrt{x_i}$. If the auctioneer runs the uniform price auction, it can be shown that the price rule is

$$p^*(\theta_i, \theta_j) = \sqrt{\theta_i^2 + \theta_j^2} \text{ with demand function } x_i^d = \frac{\theta_i^2}{p^2}.$$

Under this pricing rule, the agent's expected utility at interim stage will be

$$U_i(\theta_i) = \mathbb{E}(2\theta_i\sqrt{x_i^d} - p^* x_i^d) = \theta_i^2 \mathbb{E}\frac{1}{\sqrt{\theta_i^2 + \theta_j^2}} \geq 0$$

Therefore, the total expected revenue will be

$$R^p = \mathbb{E}p^*(\theta_i, \theta_j)$$

For simplicity, suppose θ_i is uniformly drawn from $[0, 1]$, then

$$R^p = \frac{1}{3}(\sqrt{2} + \arg\sinh 1) \cong 0.765.$$

If we run the auction proposed by M3, the auctioneer can charge a risk free revenue

$$\mathbb{E}\Delta = 4\mathbb{E}\theta_j - 2\mathbb{E}\sqrt{\theta_i^2 + \theta_j^2} = 2 - 2 * 0.765. < R^p$$

But when $\underline{\theta}$ is large enough, $\mathbb{E}\Delta > R^p$ will be possible.

5.2 Type-dependent Reservation Utility and Bilateral Trade

5.2.1 Endogeneous quantity supply

If the outside reservation utility is type-dependent, *a la* Myerson-Satterthwaite (1983), do theorem 5 and theorem 6 still hold? It depends on whether endowment is flexible or not. Theorem 5 is still true, but theorem 6 does not hold. Suppose each individual i's reservation utility is $\underline{u}_i(\theta)$, and let $S^{\#}(\theta_i, \theta_j) = S(\theta_i, \theta_j) - \sum \underline{u}_i(\theta_i)$ be the net social improvement, and $v_i^{\#}(x_i, \theta_i) = v_i(x_i, \theta_i) - \frac{1}{2}\sum \underline{u}_i(\theta_i)$ be the net utility improvement of individual i. Therefore, the payment rule will be based on $v_i^{\#}(x_i, \theta_i)$. Under mechanism M2, the payment rule can be modified as follows:

$$M_i^{1\#}(\hat{\theta}_i, \hat{\theta}_j) = \begin{cases} -S_{-i}^{\#}(\hat{\theta}_i, \hat{\theta}_j) + \beta^{1\#}(\hat{\theta}_j) + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)) & \text{if } \hat{\theta}_i > \hat{\theta}_j \\ \frac{1}{2}\sum[S(\hat{\theta}_i, \hat{\theta}_{-i}) - \beta^{1\#}(\hat{\theta}_i)] & \text{if } \hat{\theta}_i = \hat{\theta}_j \\ v_i^{\#}(x_i(\hat{\theta}_i, \hat{\theta}_j), \hat{\theta}_i) - \beta^{1\#}(\hat{\theta}_i) - \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)) & \text{if } \hat{\theta}_i < \hat{\theta}_j \end{cases}$$

where

$$\begin{aligned} \beta^{1\#}(\theta) &= \frac{\int_{\theta}^{\bar{\theta}}(1 - F(\tau))(S^{\#}(\tau, \tau) - h^{\#}(\tau))dF(\tau)}{[1 - F(\theta)]^2} + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)) \\ &= \beta^1(\theta) - \frac{1}{2}\sum \underline{u}_i(\theta) + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)). \end{aligned}$$

The following proposition justifies that M2 is still ex post individually rational.

Proposition 6 *If the reservation utility is type-dependent $\underline{u}_i(\theta_i)$, when n=2, M2 with k=1 is still ex post socially efficient if and only if $\mathbb{E}\Delta \geq \sum \underline{u}_i(\theta_i^{\#})$, where $\theta_i^{\#} = \arg\min_{\theta_i} E_{\theta_j} S(\theta_i, \theta_j) - \underline{u}_i(\theta_i)$.*

Proof. It can be shown that

$$\begin{aligned} \beta^{1\#}(\theta) &= \frac{\int_{\theta}^{\bar{\theta}}(1 - F(\tau))(S^{\#}(\tau, \tau) - h^{\#}(\tau))dF(\tau)}{[1 - F(\theta)]^2} + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)) \\ &= \frac{\int_{\theta}^{\bar{\theta}}(1 - F(\tau))(S(\tau, \tau) - h(\tau))dF(\tau)}{[1 - F(\theta)]^2} - \underline{u}(\theta) + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)) \\ &\leq \frac{1}{2}S(\theta, \theta) - \frac{1}{2}\sum \underline{u}_i(\theta) + \frac{1}{2}(\underline{u}_i(\underline{\theta}_i) - \underline{u}_j(\underline{\theta}_j)). \end{aligned}$$

Note that $\frac{d}{d\theta}\left(\frac{1}{2}S(\theta,\theta)-\frac{1}{2}\sum \underline{u}_i(\theta)\right)\geq 0$, and $\beta^{1\#}(\bar{\theta})=\frac{1}{2}S(\bar{\theta},\bar{\theta})-\frac{1}{2}\sum \underline{u}_i(\bar{\theta})$, therefore, $\beta^{1\#}(\theta)$ will be monotonically increasing in θ. We want to show the following two inequalities:

$$\begin{aligned} S(\theta_i,\theta_j)-\frac{1}{2}(\underline{u}_i(\theta_i)+\underline{u}_j(\theta_i))-\beta^{1\#}(\theta_i) &\geq \underline{u}_j(\theta_j) \text{ for } \theta_i\leq\theta_j \\ \frac{1}{2}(\underline{u}_i(\theta_i)+\underline{u}_j(\theta_i))+\beta^{1\#}(\theta_i) &\geq \underline{u}_i(\theta_i) \text{ for } \theta_i\leq\theta_j \end{aligned}$$

For the first inequality, by the supermodularity of $S(\theta_i,\theta_j)$, we have

$$\begin{aligned} & S(\theta_i,\theta_j)-\frac{1}{2}(\underline{u}_i(\theta_i)+\underline{u}_j(\theta_i))-\beta^{1\#}(\theta_i)-\underline{u}_j(\theta_j)-\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j)) \\ \geq\ & S(\theta_i,\theta_j)-\underline{u}_j(\theta_j)-\frac{1}{2}S(\theta_i,\theta_i)-\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j)) \\ \geq\ & \frac{1}{2}S(\theta_i,\theta_i)-\underline{u}_j(\theta_i)-\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j)) \\ \geq\ & \frac{1}{2}S(\underline{\theta}_j,\underline{\theta}_j)-\underline{u}_j(\underline{\theta}_j)-\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j)) \\ \geq\ & 0 \end{aligned}$$

For the second inequality, note that by the proof of theorem 5,

$$\frac{1}{2}\sum \underline{u}_i(\theta_i)+\beta^{1\#}(\theta_i)-\underline{u}_i(\theta_i)=\beta^1(\theta_i)-\underline{u}_i(\theta_i)+\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j))$$

will increase with θ_i due to $\frac{1}{2}\frac{dS(\theta_i,\theta_i)}{d\theta_i}\geq \underline{u}_i'(\theta_i)$, therefore it sufficies to show,

$$\beta^{1\#}(\underline{\theta})=\frac{1}{2}\mathbb{E}\Delta-\underline{u}_i(\underline{\theta})+\frac{1}{2}(\underline{u}_i(\underline{\theta}_i)-\underline{u}_j(\underline{\theta}_j))=\frac{1}{2}[\mathbb{E}\Delta-\sum \underline{u}_i(\underline{\theta})]\geq 0.$$

Note that $\underline{\theta}=\arg\min_{\theta_i}\{\mathbb{E}_{\theta_j}S(\theta_i,\theta_j)-\underline{u}_i(\theta_i)\}$, therefore, $\beta^{1\#}(\underline{\theta})\geq 0$ if and only if $\mathbb{E}\Delta-\sum \underline{u}_i(\underline{\theta})\geq 0$. Q.E.D. ■

The above justification applies to the trade problem where the quantity is endogeneous. Unfortunately, this conclusion may not hold for an endowment economy, as we will see below.

5.2.2 Fixed quantity supply

Suppose the quantity of supply is fixed at $\sum \bar{x}_i=1$, and let $\underline{u}_i(\theta)=v(\bar{x}_i,\theta)$ be the alternative option value of trade. According to Krishna and Perry (1998) or Palfrey and Ledyard (2007), there exist interim socially efficient trade mechanisms if and only if $\mathbb{E}\Delta-\sum \underline{u}_i(\underline{\theta}_i^{\#})\geq 0$, where $\underline{\theta}_i^{\#}=\arg\min_{\theta_i}\mathbb{E}_{\theta_j}S(\theta_i,\theta_j)-\underline{u}_i(\theta_i)$. So far we know that the classic Myerson-Satterwaite scenario fails this condition when preference is linear, initial endowment is extreme, and the quantity is indivisible. Any change of one of these three conditions may cause differences of consequence. For example, $\mathbb{E}\Delta-\sum \underline{u}_i(\underline{\theta}_i^{\#})\geq 0$ happens when initial endowment is fairly symmetric (Crampton, Gibbons and Klemperer, 1986), even if utility is still linear[13]. Furthermore, if the agent's utility is concave, then existences appear when the initial endowment is either fairly symmetric or extreme[14], as we will see below. However, once we take the ex post IR into consideration, the results are significantly changed. The following propostion describes several impossibility results.

Proposition 7 *If utility is identical $v(x_i, \theta_i)$ with i.i.d. θ_i, and total endowment is fixed, in the any of following situations, there does not exist an ex post socially efficient mechanism:*

(i) the utility $v(x_i, \theta_i)$ is linear $\theta_i x_i$, for any initial endowment allocation;

(ii) the lowest type $v(x_i, \underline{\theta}) = 0$, for any $v(x_i, \theta)$ and extreme initial endowment allocation. **(Proof see Appendix A6)**

Part (i) says that the possibility of trade will dispear again once ex post IR is considered in a partner dissolving game where each bidder initially owns some lottery or share of the object. Part (ii) indicates that the lowest type agent's value plays some subtle role, interacting with the endowment allocation. If $v(x_i, \underline{\theta}) = 0$, the designer is not able to punish the lowest type agent, therefore whole incentive scheme will be affected.

To detect the existence of an ex post socially efficient mechanism, we provide the following sufficient condition based on M3..

Proposition 8 *If the utility is symmetric and quasi-linear $v(x_i, \theta_i) = \theta_i \phi(x_i)$, there exist an ex post socially efficient mechanism if one of the traders' initial endowment satisfies the following conditions:*

$$\phi^{-1}\left[2\phi(\frac{1}{2}) - \frac{r^1(\theta_i)}{\theta_i}\right] \geq \bar{x}_i \geq \frac{1}{2} \tag{11}$$

where $r^1(\theta_i)$ is defined as formula (11). ***(Proof see Appendix A8)***

Proof. Based on M3, ex post IR requires two inequalities

$$\begin{aligned} S(\theta_i, \theta_j) - r^1(\theta_j) &\geq \underline{u}_i(\theta_i) \text{ if } \theta_i \leq \theta_j \\ r^1(\theta_j) &\geq \underline{u}_j(\theta_j) \text{ if } \theta_i \leq \theta_j \end{aligned}$$

[13] MacAfee (1992) shows that if endowment is uncertain, then there exists an interim socially efficient mechanism too.

[14] When initial endownment is extremely distributed, there exists an interim socially efficient trade mechanism if and only if $\mathbb{E}\Delta + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial}{\partial \tau} v(x_S^*(\tau, \theta_j), \tau) d\tau dF(\theta_j) \geq v(1, \bar{\theta})$.

Note that $S(\theta_i, \theta_j) - \underline{u}_i(\theta_i)$ decreases with θ_i, the first inequality holds if and only if $S(\theta_j, \theta_j) - r^1(\theta_j) \geq \underline{u}_i(\theta_j)$; and the second inequality holds if and only if $\frac{1}{2}S(\theta_j, \theta_j) \geq \underline{u}_j(\theta_j)$. Substituting $x^*(\theta_j, \theta_j) = \frac{1}{2}$ into these inequalitis, we obtain (11). Q.E.D. ■

The window of initial endowment allocation needs to meet two conditions; on one hand, it should not too asymmetric to meet interim IR constraint, on the other hand, it should not be too symmetric, allowing ex post IR to hold. The following explicit example demonstrates the intuition.

Example 5 *From example 4, under M3, ex post IR requires two inequalities*

$$\begin{aligned} S(\theta_i, \theta_j) - r^1(\theta_j) &\geq \underline{u}_i(\theta_i) \text{ if } \theta_i \leq \theta_j \\ r^1(\theta_j) &\geq \underline{u}_j(\theta_j) \text{ if } \theta_i \leq \theta_j \end{aligned}$$

which requires

$$\theta_i + \theta_j - \frac{\theta_i \theta_j}{\theta_i + \theta_j} - \frac{\ln 4 + 1}{3}\theta_j \geq \theta_i(2x_i - x_i^2)$$

The necessary and sufficient condition is

$$\begin{aligned} \frac{3}{2} - \frac{\ln 4 + 1}{3} &\geq (2x_i - x_i^2) \\ \frac{\ln 4 + 1}{3} &\geq (2x_j - x_j^2) = 1 - x_i^2 \end{aligned}$$

The endowment satisfies the above requirement is

$$0.456 \cong 1 - \frac{1}{6}\sqrt{6}\sqrt{2\ln 4 - 1} \geq x_i \geq \sqrt{1 - \frac{\ln 4 + 1}{3}} \cong 0.452$$

6 Conclusion and discussion

The basic findings of the present paper can be summarized as follows. First, in private good environments, we prove that the existence of an ex post IR socially efficient mechanism if and only if the VCG mechanism runs expected social surplus, which is the same as the condition for the existence of interim socially efficient mechanisms. Interestingly, we prove that our mechanism is the generically unique Bayesian mechanism satisfying ex post budget balance, ex post IR and ex post payoff monotonicity, which maximizes the probability of ex post IC. Our mechanism can be implemented through a specific auction for social surplus, which can be regarded as an auction for an entilement associating with post-auction redistributions. Compared with standard auctions, this auction enables the seller to earn a risk-free revenue, and the bidders to be ex post individually rational. It is also worth pointing out that we are able to characterize the bidding strategy explicitly even if the distribution is asymmetric, which is in general hard to solve in a standard first price auction.

Second, in public good environments, we find the flexibility of supply matters. If the supply of quantity is flexible, and when the number of agent is $n = 2$, there exist ex post efficient mechanisms whenever the interim socially efficient mechanisms exist, it does depend on whether or not the IR constraint is type-dependent. If the supply of quantity is fixed like a mutli-unit auction or a divisible good auction, the conclusion is only true for a type-independent outside reservation. In the fixed quantity case, our proposed auction generates a risk-free revenue the same as the expected social surplus of a VCG, which is always efficient, but the seller's revenue might be higher or lower than the uniform price auction. When the IR constraint is type-dependent like a bilaterial trade, there does not exist an ex post socially efficient mechanism even though its interim counterpart exists. For example, there does not exists an ex post socially efficient partner dissolving mechanism even though the initial endowment is symmetric, in contrast to Crampton, Klemperer and Gibbons (1986). We show non-existence of any ex post socially efficient trade if either utility is linear or the lowest type agent gain zero utility with an extreme initial endowment allocation. We also provide a sufficient condition for the existence of ex post socially efficient mechanims and an explicit example. This observation is helpful in understanding no-trade possibility in a more general context, where the trader's preference and the distribution of initial endowment have been taken into consideration.

Last but not least, we raise a concrete example where the number of players matters in determing the existence of an ex post individually rational mechanism. Due to the existence of externality, when $n > 2$, it will be more expensive to fully incorporate externatities as the numbers of players gets larger. Therefore the punishment of low boundary type will be too heavy (or the rewards to the upper bound type are too high), which may break ex post IR. There are several important extensions, including inter-dependent value, affiliation, correlated type or information acquisition in our prescribed auction, but we leave them for future discovery.

7 Appendix

7.1 A1. Proof of Proposition 1

Proof. (i) Note that

$$\begin{aligned} v_i(x_i^*(\theta_i,\theta_{-i}),\theta_i)+S_{-i}(\theta_i,\theta_{-i}) &\geq v_i(x_i^*(\theta_i',\theta_{-i}),\theta_i)+S_{-i}(\theta_i',\theta_{-i}) \\ v_i(x_i^*(\theta_i',\theta_{-i}),\theta_i')+S_{-i}(\theta_i',\theta_{-i}) &\geq v_i(x_i^*(\theta_i,\theta_{-i}),\theta_i')+S_{-i}(\theta_i,\theta_{-i}) \end{aligned}$$

therefore, $x_i^*(\theta_i,\theta_{-i})$ is non-decreasing in θ_i due to the supermodularity of $v_i(.,\theta_i)$. Moreover, note that,

$$\begin{aligned} & v_i(x_i^*(\theta_i,\theta_j,\theta_{-ij}),\theta_i)+\sum_{k\neq i} v_k(x_k^*(.),\theta_k)-c(\mathbf{x}^*(\theta_i,\theta_j,\theta_{-ij})) \\ \geq\ & v_i(x_i^*(\theta_i,\theta_j',\theta_{-ij}),\theta_i)+\sum_{k\neq i} v_k(x_k^*(.),\theta_k)-c(x_i^*(\theta_i,\theta_j',\theta_{-ij}),x_j^*(\theta_i,\theta_j,\theta_{-ij}),x_{-ij}^*) \end{aligned}$$

and

$$\begin{aligned} & v_i(x_i^*(\theta_i,\theta_j',\theta_{-ij}),\theta_i)+\sum_{k\neq i} v_k(x_k^*(.),\theta_k)-c(\mathbf{x}^*(\theta_i,\theta_j',\theta_{-ij})) \\ \geq\ & v_i(x_i^*(\theta_i,\theta_j,\theta_{-ij}),\theta_i)+\sum_{k\neq i} v_k(x_k^*(.),\theta_k)-c(x_i^*(\theta_i,\theta_j,\theta_{-ij}),x_j^*(\theta_i,\theta_j,\theta_{-ij}),x_{-ij}^*) \end{aligned}$$

we have,

$$\begin{aligned} & c(x_i^*(\theta_i,\theta_j,\theta_{-ij}),x_j^*(\theta_i,\theta_j,\theta_{-ij}),x_{-ij}^*)+c(x_i^*(\theta_i,\theta_j',\theta_{-ij}),x_j^*(\theta_i,\theta_j',\theta_{-ij}),x_{-ij}^*) \\ \leq\ & c(x_i^*(\theta_i,\theta_j',\theta_{-ij}),x_j^*(\theta_i,\theta_j,\theta_{-ij}),x_{-ij}^*)+c(x_i^*(\theta_i,\theta_j,\theta_{-ij}),x_j^*(\theta_i,\theta_j',\theta_{-ij}),x_{-ij}^*) \end{aligned}$$

If $\theta_j'\geq\theta_j$, $x_j^*(\theta_i,\theta_j',\theta_{-ij})\geq x_j^*(\theta_i,\theta_j,\theta_{-ij})$, the above inequality implies $x_i^*(\theta_i,\theta_j',\theta_{-ij})\geq x_i^*(\theta_i,\theta_j,\theta_{-ij})$ by the submodularity of $c(\mathbf{x})$. The same logic applies to $\theta_j'\leq\theta_j$. Therefore, $S(\theta_i,\theta_j,\theta_{-ij})$ is supermodular.

(ii) We can derive the similar property based on the Lagrangian. Note that

$$\begin{aligned} & v_i(x_i^*(\theta_i,\theta_{-i}),\theta_i)+v_j(x_j^*(\theta_i,\theta_j,\theta_{-ij}),\theta_j) \\ & +\sum_{k\neq i,j} v_k(x_k^*(.),\theta_k)+\lambda^*(\theta_i,\theta_j,\theta_{-ij})[\bar{c}-c(\mathbf{x}^*(\theta_i,\theta_j,\theta_{-ij}))] \\ \geq\ & v_i(x_i^*(\theta_i',\theta_{-i}),\theta_i)+v_j(x_j^*(\theta_i',\theta_j,\theta_{-ij}),\theta_j) \\ & +\sum_{k\neq i,j} v_k(x_k^*(.),\theta_k)+\lambda^*(\theta_i,\theta_j,\theta_{-ij})[\bar{c}-c(x_i^*(\theta_i',\theta_j,\theta_{-ij}),x_j^*(\theta_i',\theta_j,\theta_{-ij}),x_{-ij}^*)] \end{aligned}$$

and

$$
\begin{aligned}
& v_i(x_i^*(\theta_i', \theta_{-i}), \theta_i') + v_j(x_j^*(\theta_i', \theta_j, \theta_{-ij}), \theta_j) \\
& + \sum_{k \neq i,j} v_k(x_k^*(.), \theta_k) + \lambda^*(\theta_i', \theta_j, \theta_{-ij})[\bar{c} - c(\mathbf{x}^*(\theta_i', \theta_j, \theta_{-ij}))] \\
\geq \ & v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i') + v_j(x_j^*(\theta_i, \theta_j, \theta_{-ij}), \theta_j) \\
& + \sum_{k \neq i,j} v_k(x_k^*(.), \theta_k) + \lambda^*(\theta_i, \theta_j, \theta_{-ij})[\bar{c} - c(x_i^*(\theta_i, \theta_j, \theta_{-ij}), x_j^*(\theta_i, \theta_j, \theta_{-ij}), x_{-ij}^*)]
\end{aligned}
$$

Because the constraint is still binding, then we obtain inequality

$$
v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) + v_i(x_i^*(\theta_i', \theta_{-i}), \theta_i') \geq v_i(x_i^*(\theta_i', \theta_{-i}), \theta_i) + v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i')
$$

which implies that $x_i^*(\theta_i, \theta_{-i})$ increases with θ_i. At the same time, note that when the quantity is constrained, if $x_j^*(\theta_i, \theta_j, \theta_{-ij}) > x_j^*(\theta_i, \theta_j', \theta_{-ij})$, there must be at least for some k, $x_k^*(\theta_i, \theta_j, \theta_{-ij}) < x_k^*(\theta_i, \theta_j', \theta_{-ij})$. Therefore, we have

$$
\begin{aligned}
\frac{\partial}{\partial x_i} v_i(x_i^*(\theta_i, \theta_j', \theta_{-ij}), \theta_i) &= \frac{\partial}{\partial x_k} v_k(x_k^*(\theta_i, \theta_j', \theta_{-ij}), \theta_k) \\
&\geq \frac{\partial}{\partial x_k} v_k(x_k^*(\theta_i, \theta_{-i}), \theta_k) = \frac{\partial}{\partial x_k} v_k(x_k^*(\theta_i, \theta_{-i}), \theta_k)
\end{aligned}
$$

implying $x_i^*(\theta_i, \theta_j, \theta_{-ij}) \geq x_i^*(\theta_i, \theta_j', \theta_{-ij})$. Futhermore, $S(\theta_i, \theta_j, \theta_{-ij})$ is submodular. Q.E.D. ■

7.2 A2. Proof of Lemma 3.

Proof. By contradiction. Suppose that there is an incentive compatible and budget balance mechanism M^* resulting in

$$
\sum U_i^*(\underline{\theta}_i) > \mathbb{E}\Delta
$$

We can rewrite $\mathbb{E}\Delta$ as follows:

$$
\begin{aligned}
\mathbb{E}\Delta &= \int_{\boldsymbol{\theta}} S(\boldsymbol{\theta}) d\mathbf{F}(\boldsymbol{\theta}) - \sum \int_{\theta_{-i}} \left(\int_{\underline{\theta}_i}^{\bar{\theta}_i} (1 - F_i(\theta_i)) \frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i \right) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \\
&= \int_{\boldsymbol{\theta}} S(\boldsymbol{\theta}) d\mathbf{F}(\boldsymbol{\theta}) - \sum \int_{\boldsymbol{\theta}} \frac{1}{\lambda_i(\theta_i)} \frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} d\mathbf{F}(\boldsymbol{\theta})
\end{aligned}
$$

where $\frac{1-F_i(\theta_i)}{f_i(\theta_i)} = \frac{1}{\lambda_i(\theta_i)}$. If x_i is continuously differentiable so that $x_i^*(z_i, \theta_{-i})$ is an interior point, then

$$
\frac{\partial x_i^*(z_i, \theta_{-i})}{\partial z_i} \left[\sum \frac{\partial v_i(x_i^*(z_i, \theta_{-i}), \theta_i)}{\partial x_i} - \frac{\partial}{\partial x_i} c(\mathbf{x}^*(z_i, \theta_{-i})) \right] = 0
$$

If $x_i^*(z_i, \theta_{-i})$ is not continuous, say, a discrete variable, then $x_i^*(z_i, \theta_{-i})$ is a step function specified by (z_i, θ_{-i}), therefore, under every interval that $x_i^*(z_i, \theta_{-i})$ is being applied,

$$\frac{\partial}{\partial z_i}\left[\sum \frac{\partial v_i(x_i^*(z_i, \theta_{-i}), \theta_i)}{\partial x_i} - \frac{\partial}{\partial x_i} c(\mathbf{x}^*(z_i, \theta_{-i}))\right] = 0$$

Therefore, in any case $\frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} = \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}$ is true. Meanwhile, if $\mathbf{x}$ is exogeneously given, we write $S(\boldsymbol{\theta}) = \max_{\mathbf{x}, \boldsymbol{\lambda}} \sum v_i(x_i, \theta_i) + \lambda[\bar{c} - c(\mathbf{x})]$, it will also be the case $\frac{\partial S(\theta_i, \theta_{-i})}{\partial \theta_i} = \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}$ since $\bar{c} = c(\mathbf{x})$ is binding at the optimum. Moreover, note that

$$m_i(\tau_i) = m_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\tau_i} \left[\frac{\partial}{\partial z_i} \int_{\theta_{-i}} v_i(x_i^*(z_i, \theta_{-i}), \theta_i) f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i}\right]_{z_i = \theta_i} d\theta_i$$

And integrating by parts, we have

$$\begin{aligned}
\mathbb{E}m_i(\theta_i) &= m_i(\underline{\theta}_i) - \mathbb{E}_{\theta_{-i}} v_i(x_i^*(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\bar{\theta}_i} \int_{\theta_{-i}} v_i(x_i^*(\tau_i, \theta_{-i}), \tau_i) f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i} dF_i(\theta_i) \\
& \quad - \int_{\underline{\theta}_i}^{\bar{\theta}_i} \frac{1}{\lambda_i(\theta_i)} \left(\int_{\theta_{-i}} \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i} f_{-i}(\boldsymbol{\theta}_{-i}) d\boldsymbol{\theta}_{-i}\right) dF_i(\theta_i)
\end{aligned}$$

Therefore, the total money collected under mechanism M^* is

$$\begin{aligned}
\sum \mathbb{E}m_i(\theta_i) &= -\sum U_i(\underline{\theta}_i) + \sum \mathbb{E}_{\boldsymbol{\theta}}[v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) - \frac{1}{\lambda_i(\theta_i)} \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}] \\
&< -\mathbb{E}\Delta + \sum \mathbb{E}_{\boldsymbol{\theta}}[v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i) - \frac{1}{\lambda_i(\theta_i)} \frac{\partial v_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)}{\partial \theta_i}] \\
&= \mathbb{E}c(\mathbf{x}^*(\boldsymbol{\theta}))
\end{aligned}$$

This means under M^*, VCG must runs expected deficit. Given this fact, we may construct another mechanism $M^{\#}$ to subsidize each individual to make $M^{\#}$ individually rational (but not budget balanced), i.e.,

$$U_i^{\#}(\underline{\theta}_i) = U_i^*(\underline{\theta}_i) + k_i = \underline{u}_i(\underline{\theta}_i)$$

The total subsidy is

$$\sum k_i = \sum \underline{u}_i(\underline{\theta}_i) - \sum U_i^*(\underline{\theta}_i) < \sum \underline{u}_i(\underline{\theta}_i) - \mathbb{E}\Delta$$

which contridicts lemma 1, given that $|\mathbb{E}\Delta - \sum \underline{u}_i(\theta_i)|$ is the smallest amount of subsidy to make an incentive compatible mechanism to be interim socially efficient, since VCG maximizing the payment within incentive compatible mechanisms. Putting differently, when $\sum k_i = 0$, M^* is an interim socially efficient mechanism, but $\sum \underline{u}_i(\underline{\theta}_i) - \mathbb{E}\Delta > 0$ suggests that there does not exist any interim socially efficient mechanism, a contradiction. Q.E.D. ■

7.3 A3. Proof of Theorem 3

Proof. Step 1: payment only depends on the winner's type $\theta^{n:n}$

By contradiction. Suppose in general the payment function $M(\theta_1,\theta_2,...,\theta_n)$ depends on the ordered type, $\theta_1 \geq \theta_2 \geq ... \geq \theta_n$. Here, $M(\theta_1,\theta_2,...,\theta_n)$ need not to be differentiable, but $M(\theta_1,\theta_2,...,\theta_n)$ must be integrable so that $\mathbb{E}_{\theta_{-i}}[M(\theta,\theta_{-i})\backslash\theta > \max_{j\neq 1}\theta_j]$ is differentiable.

We use $f^{(n)}(\theta)$ to denote the joint pdf of all n order statistics as

$$f^{(n)}(\boldsymbol{\theta}^{(n)}) \equiv f_{1,2....,n:n}(\theta_1,....,\theta_n) = n!\prod_{i=1}^{n} f(\theta_i) \quad \bar{\theta} \geq \theta_1 \geq \theta_2 \geq ... \geq \theta_n \geq \underline{\theta}$$

Note that the efficient allocation must enable the highest type agent to win and pay, then the expected payment is,

$$\begin{aligned}
& m_i(\theta) \\
= \; & \Pr(\theta > \max_{j\neq i}\theta_j)\mathbb{E}_{\theta_{-i}}[M(\theta,\theta_{-i})\backslash\theta > \max_{j\neq 1}\theta_j] \\
& - \frac{\Pr(\theta < \max_{j\neq i}\theta_j)\mathbb{E}_{\theta_{-i}}\left[\mathbb{E}_{\theta_{-ij}}\sum_{j=0}^{n-2} M(\theta_1,...,\theta,...,\theta_n)\backslash\theta = \theta_{n-j-2:n-2})\backslash\theta < \max_{j\neq i}\theta_j\right]}{n-1} \\
= \; & \int_{\underline{\theta}}^{\theta}\int_{\underline{\theta}}^{\theta_2}...\int_{\underline{\theta}}^{\theta_{n-1}} M(\theta,\theta_{-i})f^{(n-1)}(\boldsymbol{\theta}^{(n-1)})d\theta_{-1} - \frac{1}{n-1}R(\theta)
\end{aligned}$$

where

$$R(\theta) = \int_{\theta}^{\bar{\theta}}\{\sum_{j=1}^{n-1}\underbrace{\int_{\theta}^{\theta_1}...\int_{\theta}^{\theta_{j-1}}}_{j-1}\underbrace{\int_{\underline{\theta}}^{\theta}\{\int_{\underline{\theta}}^{\theta_{j+2}}\int_{\underline{\theta}}^{\theta_{j+3}}...\int_{\underline{\theta}}^{\theta_{n-1}}}_{n-j-1} M(\theta_1,..,\theta_j,\theta,\theta_{j+2},.,\theta_n)f^{(n-1)}...\}d\theta_2\}d(\theta_1)$$

Note that $M(\boldsymbol{\theta})$ is non-decreasing in each argument, we have

$$\begin{aligned}
R(\theta) \;\; & \leq \;\; \int_{\theta}^{\bar{\theta}}\{\sum_{j=1}^{n-1}\frac{(n-1)!F(\theta)^{n-j-1}}{(n-j-1)!}\underbrace{\int_{\theta}^{\theta_1}...\int_{\theta}^{\theta_{j-1}}}_{j-1} M(\theta_1,.,\theta_j,\theta,\theta,.,\theta)f(\theta_{j-1})...\}d\theta_2\}d\theta_1 \\
& = \;\; \int_{\theta}^{\bar{\theta}}\{\sum_{j=1}^{n-1}\frac{(n-1)!F(\theta)^{n-j-1}(F(\theta_1)-F(\theta))^{j-1}}{(j-1)!(n-j-1)!}M(\underbrace{\theta_1,.,\theta_1}_{j},\theta,\theta,.,\theta)f(\theta_1)d\theta_1 \\
& \leq \;\; (n-1)\int_{\theta}^{\bar{\theta}} F(\theta_1)^{n-2}M(\underbrace{\theta_1,.,\theta_1,\theta_1}_{n-2},\theta,\theta)f(\theta_1)d\theta_1
\end{aligned}$$

Therefore,

$$m_i(\theta) \leq \int_{\underline{\theta}}^{\theta}\int_{\underline{\theta}}^{\theta_2}...\int_{\underline{\theta}}^{\theta_{n-1}} M(\theta,\theta_{-1})f^{(n-1)}(\boldsymbol{\theta}^{(n-1)})d\theta_{-1} - \int_{\theta}^{\bar{\theta}} F(\theta_1)^{n-2}M(\underbrace{\theta_1,.,\theta_1,\theta_1}_{n-2},\theta,\theta)dF(\theta_1) \tag{12}$$

Note that when $\theta \to \bar{\theta}$,

$$m_i(\bar{\theta}) = \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta_2} ... \int_{\underline{\theta}}^{\theta_{n-1}} M(\bar{\theta}, \theta_{-i}) f^{(n-1)}(\boldsymbol{\theta}^{(n-1)}) d\theta_2 ... d\theta_n$$

Since inequality (3.12) should hold for any θ, and $\bar{\theta}$, it must be the case,

$$\begin{aligned} m_i'(\theta) \quad \leq \quad & f(\theta)(n-1) \int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_3} ... \int_{\underline{\theta}}^{\theta_{n-1}} M(\theta, \theta, \theta_{-12}) f^{(n-2)}(\boldsymbol{\theta}^{(n-2)}) d\theta_{-12} \\ & + \int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_2} ... \int_{\underline{\theta}}^{\theta_{n-1}} \frac{\partial}{\partial \theta} [M(\theta, \theta_{-1})] f^{(n-1)}(\boldsymbol{\theta}^{(n-1)}) d\theta_{-1} \\ & + F(\theta)^{n-2} f(\theta) M(\theta, ..., \theta, \theta) - \int_{\theta}^{\bar{\theta}} F(\theta_1)^{n-2} \{ \frac{\partial}{\partial \theta} [M(\underbrace{\theta_1, ., \theta_1, \theta_1}_{n-2}, \theta, \theta)] \} dF(\theta_1) \end{aligned}$$

where $\frac{\partial}{\partial \theta}[M(\theta, \theta_{-1})]$ is piecewise if $M(\underbrace{\theta_1, ., \theta_1, \theta_1}_{n-2}, \theta, \theta)$ is not differentiable in θ (but $M(\underbrace{\theta_1, ., \theta_1, \theta_1}_{n-2}, \theta, \theta)$ is still weakly differentiable).

Note that $\int_{\theta}^{\bar{\theta}} F(\theta_1)^{n-2} \{ \frac{\partial}{\partial \theta} [M(\underbrace{\theta_1, ., \theta_1, \theta_1}_{n-2}, \theta, \theta)] \} dF(\theta_1) \geq 0$, therefore, we have

$$\begin{aligned} S_i(\theta) \quad \leq \quad & \frac{\int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_3} ... \int_{\underline{\theta}}^{\theta_{n-1}} M(\theta, \theta, \theta_{-12}) f^{(n-2)}(\boldsymbol{\theta}^{(n-2)}) d\theta_{-12}}{F(\theta)^{n-2}} \\ & + \frac{1}{n-1} M(\theta, ..., \theta, \theta) \\ & + \frac{F(\theta)}{f(\theta)} \frac{\int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_2} ... \int_{\underline{\theta}}^{\theta_{n-1}} \frac{\partial}{\partial \theta} \delta [M(\theta, \theta_{-1})] f^{(n-1)}(\boldsymbol{\theta}^{(n-1)}) d\theta_{-1}}{F(\theta)^{n-1}}. \end{aligned}$$

When $\theta \to \underline{\theta}$, we have $\frac{n-1}{n} S_i(\underline{\theta}) \leq M(\underline{\boldsymbol{\theta}})$. If $\frac{n-1}{n} S_i(\underline{\theta}) < M(\underline{\boldsymbol{\theta}})$, payment rule $M(\boldsymbol{\theta})$ will not lead to ex post monotonicity. So it has to be $M(\underline{\boldsymbol{\theta}}) = \frac{n-1}{n} S_i(\underline{\theta})$. Because

$$\lim_{\theta \to \underline{\theta}} \frac{F(\theta)}{f(\theta)} \frac{\int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_2} ... \int_{\underline{\theta}}^{\theta_{n-1}} \frac{\partial}{\partial \theta} \delta [M(\theta, \theta_{-1})] f^{(n-1)}(\boldsymbol{\theta}^{(n-1)}) d\theta_{-1}}{F(\theta)^{n-1}} = 0,$$

and the above inequality should hold for any θ and $\underline{\theta}$ uniformly, therefore,

$$S_i(\theta) \leq \frac{\int_{\underline{\theta}}^{\theta} \int_{\underline{\theta}}^{\theta_3} ... \int_{\underline{\theta}}^{\theta_{n-1}} M(\theta, \theta, \theta_{-12}) f^{(n-2)}(\boldsymbol{\theta}^{(n-2)}) d\theta_{-12}}{F(\theta)^{n-2}} + \frac{1}{n-1} M(\theta, ..., \theta, \theta)$$

should be true for all θ.

If $n \geq 2$, and $M(\theta, \theta, \theta_{-12})$ depends on θ_j for $j \geq 3$, we have

$$S_i(\theta) < \frac{n}{n-1} M(\theta, ..., \theta, \theta),$$

which fails ex post payoff monotonicity.

If $n = 2$, or $M(\boldsymbol{\theta}) = M(\theta_1, \theta_2)$, we have $S_i(\theta) \leq \frac{n}{n-1} M(\theta, \theta)$. Because $M(\theta, \theta) = \frac{n-1}{n} S_i(\theta)$ can not be incentive compatible, there must be $M(\theta, \theta) > \frac{n-1}{n} S_i(\theta)$ for some measurable set of θ. To see this, note that if $M(\theta, \theta) = \frac{n-1}{n} S_i(\theta)$ is incentive compatible, we will have $\frac{\partial}{\partial \theta} M(\theta, \theta) = 0$ for all θ, a contradiction. Therefore, $M(\boldsymbol{\theta})$ must depends on $\theta^{n:n}$ only.

Step 2: derivation of payment rule as a function of $\theta^{n:n}$ only.

If payment depends on $\theta^{n:n}$, we have the following integration eqution:

$$M(\theta)G(\theta) - \frac{1}{n-1} \int_{\theta}^{\bar{\theta}} M(\tau) dG(\tau) = m(\theta)$$

Note that $\int_{\theta}^{\bar{\theta}} M(\tau) dG(\tau)$ should be differentiable even if $M(\theta)$ is not differentiable. We thus can solve the above integration equation, yielding,

$$M(\theta) = \frac{n-1}{n} \frac{\int_{\underline{\theta}}^{\theta} S(\tau) dF(\tau)^n}{F(\theta)^n}.$$

which is consistent with M^F. Q.E.D. ■

7.4 A4. Proof of Proposition 4

Proof. (i) The incentive compatibility is met by construction, and so is budget balance. We only need to check ex post IR constraint. First, we need to prove the payment rule is non-negative. Observing that from the equation (6), we have

$$\begin{aligned} \sum M_i' &= -\left(\sum M_i\right)\left(\sum q_k\right) - \frac{n}{n-1}\sum q_k M_k + \frac{n}{n-1}\sum q_i M_i + S(\theta_i)(n-1)\sum q_k \\ &= -\left(\sum M_i\right)\left(\sum q_k\right) + S(\theta_i)(n-1)\sum q_k \end{aligned}$$

thus,

$$\sum M_i(\theta) = (n-1) \frac{\int_{\underline{\theta}}^{\theta} S(\tau) d\Pi(\tau)}{\Pi(\theta)} \tag{13}$$

Using this formula, we can show that if $M_i(\theta) \leq \min_{j\neq} M_j(\theta)$ for any θ, then $M_i'(\theta) > 0$. To see this, note that for any θ, if $M_i \leq \min_{j\neq} M_j$,

$$M_k = \sum M_j - \sum_{j \neq k} M_j \leq \sum M_j - \sum_{j \neq i} M_j$$

therefore

$$q_k M_k \leq q_k (\sum M_j - \sum_{j \neq i} M_j)$$

Thus,

$$\begin{aligned}
M_i' &= -M_i \sum_{k\neq i} q_k - \frac{1}{n-1} \sum_{k\neq i} q_k M_k + S(\theta_i) \sum_{k\neq i} q_k \\
&\geq -M_i \sum_{k\neq i} q_k - \frac{1}{n-1} (\sum M_j - \sum_{j\neq i} M_j) \sum_{k\neq i} q_k + S(\theta_i) \sum_{k\neq i} q_k \\
&\geq \left(\sum_{k\neq i} q_k \right) \left(S(\theta_i) - \frac{\int_{\underline{\theta}}^{\theta} S(\tau) d\Pi(\tau)}{\Pi(\theta)} \right) \\
&> 0
\end{aligned}$$

By the above inequality, payment increases with type. Meanwhile, note that $M_i(\underline{\theta}) = \frac{1}{n} S(\underline{\theta}) \geq 0$, therefore, if $M_i(\theta)$ is the lowest payment, then $M_i(\theta) > 0$; if $M_i(\theta)$ is not lowest, of course $M_i(\theta) > 0$. Therefore, the loser's pay-off is non-negative.

Now it is easy to show that the winner's ex post pay-off is also non-negative, due to

$$S(\theta_i) - M_i(\theta_i) \geq M_j(\theta_i) \geq \frac{1}{n-1} M_j(\theta_i)$$

since M_i is non-negative.

(ii) Note that,

$$\begin{aligned}
\sum U_i(\underline{\theta}) &= -\sum m_i(\underline{\theta}) \\
&= \frac{1}{n-1} \sum_{j=1} \sum_{j\neq i} \int_{\underline{\theta}}^{\bar{\theta}} M_j(z) \left[\Pi_{k\neq j,i} F_k(z_j) \right] dF_j(z) \\
&= \sum_{i=1} \mathbb{E}_{\theta_{-i}} S(\underline{\theta}_i, \theta_{-i}) - \sum_{i=1} \frac{1}{n-1} \mathbb{E}_{\theta_{-i}} \sum_{j\neq i} \mathbb{E}_{\theta_{-j}} [S_{-j}(\theta_j, \theta_{-j})] \\
&= \mathbb{E}\Delta
\end{aligned}$$

where the third step comes from plugging in equation (5). We get the conclusion. Q.E.D. ■

7.5 A5. Proof of Proposition 5

Proof. Note that for any incentive compatible mechanism, the first order condition

$$m_i'(\theta) = f_j(\theta) S_i(\theta)$$

is always the case. Thus, we obtain two equations:

$$S(\theta_i) f_j(\theta_i) = f_j M_i + F_j M_i' + M_j f_j$$

$$S(\theta_j) f_i(\theta_j) = f_i M_j + F_i M_j' + M_i f_i$$

Therefore, we have

$$0 = f_i F_j M_i' - f_j F_i M_j'$$

Taking derivative w.r.t. θ_i, we have

$$M_j' = S'(\theta_i) - M_i' - (\frac{F_j}{f_j})' M_i' - (\frac{F_j}{f_j}) M_i''$$

Therefore we obtain a second order ODE regarding a single unknown:

$$\frac{f_j}{F_j} S'(\theta_i) = M_i' \left(\frac{f_j}{F_j} + \frac{f_j}{F_j} (\frac{F_j}{f_j})' + \frac{f_i}{F_i} \right) + M_i''$$

The general solution is,

$$M_i' = \frac{f_j}{F_i F_j^2} C_1 + \frac{f_j \int_{\underline{\theta}}^{\theta} F_i F_j S' d\theta}{F_i F_j^2}$$

hence,

$$M_i = M_i(\underline{\theta}_i) + C_1 \int_{\underline{\theta}}^{\theta_i} \frac{f_j}{F_i F_j^2} d\theta + \int_{\underline{\theta}}^{\theta_i} \frac{f_j \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i F_j^2} d\theta$$

Note that in any case M_i should be non-negative and not be infinite, so $C_1 = 0$, otherwise,

$$\int_{\underline{\theta}}^{\theta_i} \frac{f_j}{F_i F_j^2} d\theta \geq \int_{\underline{\theta}}^{\theta_i} \frac{f_j}{F_j^2} d\theta = \frac{1}{F_j(\underline{\theta}_i)} - \frac{1}{F_j(\theta_i)} \to \infty.$$

It can be shown that $\int_{\underline{\theta}}^{\theta_i} \frac{f_j \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i F_j^2} d\theta$ is bounded. Even for $\theta \to \underline{\theta}$, according to L'Hospital Law,

$$\lim_{\theta \to \underline{\theta}} \frac{1}{F_j} \left(\frac{\int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i F_j} \right) = \frac{1}{2} \lim_{\theta \to \underline{\theta}} \frac{1}{f_j} S'(\underline{\theta})$$

Therefore, we have

$$\begin{aligned} M_i &= M_i(\underline{\theta}_i) + \int_{\underline{\theta}}^{\theta_i} \frac{f_j \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i F_j^2} d\theta \\ &= M_i(\underline{\theta}_i) + \frac{\int_{\underline{\theta}}^{\theta_i} S d(F_i F_j)}{F_i F_j} - \int_{\underline{\theta}}^{\theta_i} \frac{1}{F_j} \frac{\int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i^2} dF_i \end{aligned}$$

To check the incentive compatibility, we have

$$\begin{aligned}
& F_j M_i - \int_{\theta_i}^{\bar{\theta}} M_j dF_j \\
= \; & F_j \left(M_i(\underline{\theta}_i) + \int_{\underline{\theta}}^{\theta_i} \frac{f_j \int_{\underline{\theta}}^{\theta} F_i F_j S' d\theta}{F_i F_j^2} d\theta \right) - \int_{\theta_i}^{\bar{\theta}} \left(M_j(\underline{\theta}_j) + \int_{\underline{\theta}}^{\theta_j} \frac{f_i \int_{\underline{\theta}}^{\theta} F_i F_j S' d\theta}{F_j F_i^2} d\theta \right) dF_j \\
= \; & M(\underline{\theta})(2F_j - 1) - F_j(\theta_i) \left(\frac{1}{F_i F_j} \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau \Big|_{\underline{\theta}}^{\theta_i} - \int_{\underline{\theta}}^{\theta_i} S' d\theta \right) \\
& - \int_{\underline{\theta}}^{\bar{\theta}} \frac{f_i \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_j F_i^2} d\theta - \int_{\theta_i}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau d\frac{1}{F_i} \\
= \; & M(\underline{\theta})(2F_j - 1) - \int_{\underline{\theta}}^{\bar{\theta}} \frac{f_i \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_j F_i^2} d\theta - \int_{\underline{\theta}}^{\bar{\theta}} F_i F_j S' d\tau + F_j(\theta_i) \int_{\underline{\theta}}^{\theta_i} S' d\theta + \int_{\theta_i}^{\bar{\theta}} F_j S' d\theta \\
= \; & m_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} S(\theta) dF_j(\theta)
\end{aligned}$$

where

$$m(\underline{\theta}) = M(\underline{\theta})(2F_j - 1) - \int_{\underline{\theta}}^{\bar{\theta}} \frac{f_i \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_j F_i^2} d\theta - \int_{\underline{\theta}}^{\bar{\theta}} F_i F_j S' d\tau - \int_{\underline{\theta}}^{\bar{\theta}} S(\theta) dF_j(\theta) + S(\bar{\theta}) - F_j(\theta_i) S(\underline{\theta})$$

We set $M(\underline{\theta}) = \frac{1}{2} S(\underline{\theta})$, which means that for the lowest type, it is indifferent for him to lose or win. Then,

$$m_i(\underline{\theta}) = \int_{\underline{\theta}}^{\bar{\theta}} S(\theta) d(F_i F_j) - \int_{\underline{\theta}}^{\bar{\theta}} S(\theta) dF_j(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \frac{f_i \int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_j F_i^2} d\theta - \frac{1}{2} S(\underline{\theta})$$

which is a constant independent of θ.

We can verify that under this payment rule, truth-telling is an equilibrium. For an agent i to deviate from θ_i, the resulting profit difference will be:

$$\begin{aligned}
U_i(\theta_i, \theta_i) - U_i(\theta_i, \tilde{\theta}_i) &= [\Pr(\theta_j \leq \theta_i) - \Pr(\theta_j \leq \tilde{\theta}_i)] S(\theta_i) - [m_i(\theta_i) - m_i(\theta_i)] \\
&= \int_{\tilde{\theta}_i}^{\theta_i} [S(\theta_i) f_j(\tau) - m_i'(\tau)] d\tau
\end{aligned}$$

When $\theta_i > \tilde{\theta}_i$, $S(\theta_i) f_j(\tau) - m_i'(\tau) > S(\tau) f_j(\tau) - m_i'(\tau) = 0$; when $\theta_i < \tilde{\theta}_i$, $S(\theta_i) f_j(\tau) - m_i'(\tau) < S(\tau) f_j(\tau) - m_i'(\tau) = 0$, therefore, in any case, $U_i(\theta_i, \theta_i) - U_i(\theta_i, \tilde{\theta}_i) > 0$ for any $\theta_i \neq \tilde{\theta}_i$.

For the ex post monotonicity, it is easy to see that the above payment rule is monotone in θ and non-negative. We only need to verify

$$S(\theta_i) - M_i(\theta_i) \geq M_j(\theta_j) \text{ for } \theta_i \geq \theta_j.$$

by noting that

$$\begin{aligned}
& M_i(\theta) + M_j(\theta) \\
= & \; S(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{f_j \int_{\underline{\theta}}^{z} F_i F_j S' d\tau}{F_i F_j^2} d\theta + \int_{\underline{\theta}}^{\theta} \frac{f_i \int_{\underline{\theta}}^{\theta} F_j F_i S' d\tau}{F_j F_i^2} d\theta \\
= & \; S(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left(\int_{\underline{\theta}}^{z} F_i F_j S' d\tau \right) d\frac{1}{F_i F_j} \\
= & \; S(\theta) - \frac{\int_{\underline{\theta}}^{\theta} F_i F_j S' d\tau}{F_i F_j} \\
= & \; \frac{\int_{\underline{\theta}}^{\theta} S d(F_i F_j)}{F_i F_j} \leq S(\theta)
\end{aligned}$$

Q.E.D. ■

7.6 A6. Proof of Theorem 4

Proof. (i) It is obvious to see that **M2** is ex post budget balance by construction. The incentive compatibility could be checked as follows. Note that the structure of expected payment $m(\theta)$ can be written as:

$$\begin{aligned}
m(\theta) = & \; (1-k)G(\theta)\beta^k(\theta) + k\int_{\underline{\theta}}^{\theta} \beta^k(\tau) dG(\tau) - \int_{\theta_{-i} \leq \theta} S_{-i}(\theta, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i}) \\
& - \frac{1}{n-1} \left(k \left[\begin{array}{c} (n-1)F(\theta)^{n-2}(1-F(\theta))\beta^k(\theta) \\ +(n-1)\int_{\theta}^{\bar{\theta}} (1-F(\tau))\beta^k(\tau) dF(\tau)^{n-2} \end{array} \right] + (1-k)\int_{\theta}^{\bar{\theta}} \beta^k(\tau) dG(\tau) \right) \\
& + \sum_{j=2}^{n} \Pr(\theta \text{ is } j\text{-th highest order statistic}) \mathbb{E}[v(x^*(\theta, \theta_{-i}), \theta)/\theta \text{ is } j\text{-th highest order statistic}] \\
= & \; (1-k)G(\theta)\beta^k(\theta) + k\int_{\underline{\theta}}^{\theta} \beta^k(\tau) dG(\tau) - \int_{\theta_{-i} \leq \theta} S_{-i}(\theta, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i}) \\
& - k \left[F(\theta)^{n-2}(1-F(\theta))\beta^k(\theta) + \int_{\theta}^{\bar{\theta}} (1-F(\tau))\beta^k(\tau) dF(\tau)^{n-2} \right] - \frac{(1-k)}{n-1} \int_{\theta}^{\bar{\theta}} \beta^k(\tau) dG(\tau) \\
& + \sum_{j=1}^{n} \bar{v}_i^{(j)}(x_i^*(.., \theta, .), \theta) - \bar{v}_i^{(1)}(x_i^*(.., \theta, .), \theta)
\end{aligned}$$

(Here $v_i(x_i^*(..\theta_{j-1}, \theta, .\theta_{j+1}, .), \theta)$ means individual i's type is θ, which is j-th highest order statistic among n). Taking derivative w.r.t θ and simplifying the above equation, we obtain a differential equation regarding $\beta^k(\theta)$:

$$\begin{aligned}
& m'(\theta) + \frac{d}{d\theta}\left(\int_{\theta_{-i}\leq\theta} S_{-i}(\theta,\theta_{-i})f^{(n-1)}(\theta_{-i})d\theta_{-i} - \sum_{j=1}^{n}\bar{v}_i^{(j)}(x_i^*(..,\theta,.),\theta)\right) \\
= \; & (1-k)G(\theta)\beta'(\theta) - kF(\theta)^{n-2}(1-F(\theta))\beta^{k\prime}(\theta) + (1-k)g(\theta)\beta^k(\theta + kg(\theta)\beta^k(\theta) \\
& -kF(\theta)^{n-3}((n-2)-(n-1)F(\theta))\beta^k(\theta)f(\theta) \\
& +k(n-2)(1-F(\theta))\beta^k(\theta)F(\theta)^{n-3}f(\theta) + \frac{(1-k)}{n-1}g(\theta)\beta^k(\theta) \\
= \; & \beta^{k\prime}(\theta)(F(\theta)-k)F(\theta)^{n-2} + nF(\theta)^{n-2}f(\theta)\beta^k(\theta)
\end{aligned}$$

$$\begin{aligned}
= \; & (1-k)G(\theta)\beta^k(\theta) + k\int_{\underline{\theta}}^{\theta}\beta(\tau)dG(\tau) - \int_{\theta_{-i}\leq\theta} S_{-i}(\theta,\theta_{-i})f^{(n-1)}(\theta_{-i})d\theta_{-i} \\
& -k\left[F(\theta)^{n-2}(1-F(\theta))\beta^k(\theta) + \int_{\theta}^{\bar{\theta}}(1-F(\tau))\beta^k(\tau)dF(\tau)^{n-2}\right] - \frac{(1-k)}{n-1}\int_{\theta}^{\bar{\theta}}\beta^k(\tau)dG(\tau)
\end{aligned}$$

Plugging $m'(\theta)$ into the above equation, the ODE becomes

$$\begin{aligned}
& \beta^{k\prime}(\theta)(F(\theta)-k)F(\theta)^{n-2} + nF(\theta)^{n-2}f(\theta)\beta^k(\theta) \\
= \; & \frac{d}{d\theta}\left(\int_{\theta_{-i}\leq\theta} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})\right) - \frac{d}{d\theta}\sum_{j=1}^{n}\bar{v}_i^{(j)}(x_i^*(..,\theta,.),\theta) \\
& + \left[\frac{\partial}{\partial z_i}\int_{\theta_{-i}} v_i(x_i^*(z_i,\theta_{-i}),\theta)d\mathbf{F}_{-i}(\theta_{-i})\right]_{z_i=\theta}
\end{aligned}$$

The solution turns out to be

$$\beta^k(\theta) = \frac{\int_{F^{-1}(k)}^{\theta}\left(\frac{d}{d\tau}\bar{S}^{(1)}(\tau) - \sum_{j=1}^{n}\frac{d}{d\tau}\bar{v}_i^{(j)}(x_i^*(\tau,.),\tau) + m'(\tau)\right)\frac{(F(\tau)-k)^{n-1}}{F(\tau)^{n-2}}d\tau}{(F(\theta)-k)^n}$$

which means **M2** is incentive compatible. To check the second order condition, just apply the proof of lemma 3.

$$\begin{aligned}
& U(\theta,\theta) - U(\theta,\tilde{\theta}) \\
= \; & \int_{\theta_{-i}}[v(x(\theta,\theta_{-i}),\theta) - v(x(\tilde{\theta},\theta_{-i}),\theta)]dF_{-i}(\theta_{-i}) - (m(\theta) - m(\tilde{\theta})) \\
= \; & \int_{\tilde{\theta}}^{\theta}\left(\frac{\partial}{\partial z}\int_{\theta_{-i}} v(x(z,\theta_{-i}),\theta)d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i})\right)dz - \int_{\tilde{\theta}}^{\theta} m'(z)dz
\end{aligned}$$

From the IC constraint,

$$m'(z) = \left[\frac{\partial}{\partial\tau}\int_{\theta_{-i}}\frac{\partial}{\partial x}v(x(\tau,\theta_{-i}),z)d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i})\right]_{\tau=z}$$

Note that $x(z, \theta_{-i})$ is increasing function of z, and $\frac{\partial}{\partial x} v(x(z, \theta_{-i}), \theta)$ is increasing function of θ, therefore,

$$\begin{aligned}
& U(\theta, \theta) - U(\theta, \tilde{\theta}) \\
= & \int_{\tilde{\theta}}^{\theta} \left[\frac{\partial}{\partial z} \int_{\theta_{-i}} v(x(z, \theta_{-i}), \theta) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) - \left[\frac{\partial}{\partial \tau} \int_{\theta_{-i}} \frac{\partial}{\partial x} v(x(\tau, \theta_{-i}), z) d\mathbf{F}_{-i}(\boldsymbol{\theta}_{-i}) \right]_{\tau = z} \right] d\mathbf{F}_{-i}(\theta_{-i}) dz \\
\geq & \ 0
\end{aligned}$$

(ii) To show this, it is convenient to apply the revenue equivalence principle. With assistance of $\beta^0(\theta)$, we know

$$m(\underline{\theta}) = -\frac{1}{n-1} \int_{\underline{\theta}}^{\bar{\theta}} \beta^k(\theta) dG(\theta) + \int_{\theta_{-i}} v(x^*(\underline{\theta}, \theta_{-i}), \underline{\theta}) d\mathbf{F}_{-i}(\theta_{-i})$$

And the lowest type agent's payoff:

$$U(\underline{\theta}) = \mathbb{E} v(x^*(\underline{\theta}, \theta_{-i}), \underline{\theta})] - m(\underline{\theta}) = \frac{1}{n-1} \int_{\underline{\theta}}^{\bar{\theta}} \beta^0(\theta) dG(\theta)$$

Plugging the above equality into the payment function:

$$\begin{aligned}
& \frac{1}{n-1} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\int_{\underline{\theta}}^{\theta} F(\tau) d \int_{\theta_{-i} \leq \tau} S(\tau, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i})}{F(\theta)^n} dG(\theta) \\
= & -\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} F(\tau) d \int_{\theta_{-i} \leq \tau} S(\tau, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i}) \right) d \frac{1}{F(\theta)} \\
= & \int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta_{-i} \leq \theta} S(\theta, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i}) dF(\theta)
\end{aligned}$$

The third line of the above simplification comes from integrating by parts. Similarly,

$$\begin{aligned}
& -\frac{1}{n-1} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\int_{\theta_{-i}} \left(\int_{\underline{\theta}}^{\theta} F(\tau) \frac{\partial}{\partial \tau} v(x^*(\tau, \theta_{-i}), \tau) d\tau \right) d\mathbf{F}_{-i}(\theta_{-i})}{F(\theta)^n} dG(\theta) \\
= & \int_{\theta_{-i}} \left[\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} F(\tau) \frac{\partial}{\partial \tau} v(x^*(\tau, \theta_{-i}), \tau) d\tau \right) d \frac{1}{F(\theta)} \right] d\mathbf{F}_{-i}(\theta_{-i}) \\
= & -\int_{\theta_{-i}} \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{1 - F(\theta)}{f(\theta)} \frac{\partial}{\partial \theta} v(x^*(\theta, \theta_{-i}), \theta) dF(\theta) \right) d\mathbf{F}_{-i}(\theta_{-i})
\end{aligned}$$

Therefore, the lowest type agent's utility turns out to be:

$$\begin{aligned}
U(\underline{\theta}) = & \frac{1}{n-1} \int_{\underline{\theta}}^{\bar{\theta}} \beta^0(\theta) dG(\theta) \\
= & \int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta_{-i} \leq \theta} S(\theta, \theta_{-i}) d\mathbf{F}_{-i}(\theta_{-i}) dF(\theta) - \int_{\theta_{-i}} \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{1 - F(\theta)}{f(\theta)} \frac{\partial}{\partial \theta} v(x^*(\theta, \theta_{-i}), \theta) dF(\theta) \right) d\mathbf{F}_{-i}(\theta_{-i})
\end{aligned}$$

Note that[15]

$$\frac{1}{n}\int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta_{-i}} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})dF(\theta)=\int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta_{-i}\leq\theta} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})dF(\theta)$$

So $U(\underline{\theta})=\frac{1}{n}\mathbb{E}\Delta$. Q.E.D. ■

7.7 A7. Derivation of M3.

Let $\Psi(\tau)=1-(1-F(\tau))^{n-1}$ be the distribution of $Z_i=\min_{i\neq j}\theta_j$, the expected payment can be written as

$$\begin{aligned}
m(\theta) &= (1-k)(1-\Psi(\theta))r^k(\theta)+k\int_{\theta}^{\bar{\theta}} r^k(\tau)d\Psi(\tau)-\int_{\theta_{-i}\geq\theta} S_{-i}(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i}) \\
&\quad -\frac{1}{n-1}\left(k\begin{bmatrix} (n-1)F(\theta)(1-F(\theta))^{n-2}r^k(\theta) \\ +(n-1)(n-2)\int_{\underline{\theta}}^{\theta} r^k(\tau)F(\tau)(1-F(\tau))^{n-3}dF(\tau)\end{bmatrix}+(1-k)\int_{\underline{\theta}}^{\theta} r(\tau)d\Psi(\tau)\right) \\
&\quad +\sum_{j=2}^{n}\Pr(\theta\text{ is }j\text{-th smallest order statistic})\mathbb{E}[v(x^*(\boldsymbol{\theta}),\theta)/\theta\text{ is }j\text{-th smallest order statistic}] \\
&= (1-k)(1-\Psi(\theta))r^k(\theta)+k\int_{\theta}^{\bar{\theta}} r^k(\tau)d\Psi(\tau)-\int_{\theta_{-i}\geq\theta} S_{-i}(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i}) \\
&\quad -k\left[F(\theta)(1-F(\theta))^{n-2}r^k(\theta)+(n-2)\int_{\underline{\theta}}^{\theta} r^k(\tau)F(\tau)(1-F(\tau))^{n-3}dF(\tau)\right] \\
&\quad -\frac{(1-k)}{n-1}\int_{\underline{\theta}}^{\theta} r(\tau)d\Psi(\tau)+\int_{\theta_{-i}} v(x^*(\theta,\theta_{-i}),\theta)d\mathbf{F}_{-i}(\theta_{-i})-\int_{\theta_{-i}\geq\theta} v(x^*(\theta,\theta_{-i}),\theta)d\mathbf{F}_{-i}(\theta_{-i})
\end{aligned}$$

[15] Note that the following formula is true by symmetricity of $S(\theta,\theta_{-i})$:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta_{-i}} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})dF(\theta) \\
=& \sum_{i=1}\Pr(i\text{ is the i-th highest order})\mathbb{E}S(\theta_i,\theta_{-i})/i\text{ is the i-th highest order statistics}] \\
=& \mathbb{E}S(\theta_i,\theta_{-i})/i\text{ is the i-th highest order statistics}] \\
=& \int_{\underline{\theta}}^{\bar{\theta}}\frac{\int_{\theta_{-i}\leq\theta} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})}{F(\theta)^{n-1}}dF(\theta)^n \\
=& n\int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta_{-i}\leq\theta} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})dF(\theta)
\end{aligned}$$

The above equality can also come from Fubini's theorem.

Taking derivative w.r.t θ to simplify the above equition, we obtain a differential equation regarding $r^k(\theta)$:

$$\begin{aligned}
& m'(\theta) + \frac{d}{d\theta}\left(\int_{\theta_{-i}\geq\theta} S(\theta,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i}) - \sum_{j=1}^{n}\frac{d}{d\tau}\bar{v}_i^{(j)}(x_i^*(\theta,.),\theta)\right) \\
= \ & (1-k)(1-\Psi(\theta))r^{k\prime}(\theta) - (1-k)\Psi'(\theta)r^k(\theta) - kr^k(\theta)\Psi'(\theta) \\
& -k\left[\begin{array}{c} F(\theta)(1-F(\theta))^{n-2}r^{k\prime}(\theta) - (n-2)F(\theta)(1-F(\theta))^{n-3}f(\theta)r^k(\theta) \\ +(1-F(\theta))^{n-2}f(\theta)r^k(\theta) + (n-2)(1-F(\theta))^{n-3}F(\theta)f(\theta)r^k(\theta)] \end{array}\right] \\
& -\frac{(1-k)}{n-1}r^k(\theta)\Psi'(\theta) \\
= \ & r^{k\prime}(\theta)[(1-k)(1-\Psi(\theta)) - kF(\theta)(1-F(\theta))^{n-2}] \\
& -r^k(\theta)[\Psi'(\theta) + k(1-F(\theta))^{n-2}f(\theta) + \frac{(1-k)}{n-1}\Psi'(\theta)] \\
= \ & r^{k\prime}(\theta)[(1-k-F(\theta)](1-F(\theta))^{n-2} - r^k(\theta)n(1-F(\theta))^{n-2}f(\theta)
\end{aligned}$$

The solution for this ODE is

$$r^k(\theta) = \frac{\int_{F^{-1}(1-k)}^{\theta}\frac{[k-1+F(\tau)]^{n-1}}{(1-F(\tau))^{n-2}}(\sum_{j=1}^{n}\frac{d}{d\tau}\bar{v}_i^{(j)}(x_i^*(\tau,.),\tau) - m'(\tau) - \frac{d}{d\tau}\bar{S}^{(n)}(\tau))d\tau}{[k-1+F(\theta)]^n}$$

Thus, if $k=0$,

$$\begin{aligned}
U(\underline{\theta}) &= \mathbb{E}S(\underline{\theta},\theta_{-i}) - r^0(\underline{\theta}) \\
&= \mathbb{E}S(\underline{\theta},\theta_{-i}) - \int_{\underline{\theta}}^{\bar{\theta}}(1-F(\tau))\left(\int_{\theta_{-i}}\frac{\partial}{\partial\tau}v(x^*(\tau,\theta_{-i}),\tau)d\mathbf{F}_{-i}(\theta_{-i}) - \frac{d}{d\tau}\int_{\theta_{-i}\geq\tau}S(\tau,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i})\right)d\tau \\
&= \int_{\underline{\theta}}^{\bar{\theta}}\int_{\theta_{-i}\geq\tau}S(\tau,\theta_{-i})d\mathbf{F}_{-i}(\theta_{-i}))dF(\tau) - \int_{\theta_{-i}}\int_{\underline{\theta}}^{\bar{\theta}}\frac{1}{\lambda(\tau)}\frac{\partial}{\partial\tau}v(x^*(\tau,\theta_{-i}),\tau)dF(\tau)d\mathbf{F}_{-i}(\theta_{-i}) \\
&= \mathbb{E}\Delta
\end{aligned}$$

7.8 A8. Proof of Proposition 7

Proof. (i) Let q_i be the initial endowment and $\theta_i^* = F^{-1}(q_i)$. Without loss of generality, suppose $q_j \geq q_i$, therefore $\theta_j^* \geq \theta_i^*$, thus $U_i(\theta_i^*) - \theta_i^* q_i \geq U_j(\theta_j^*) - \theta_j^* q_j$. Ex post IR requires the following conditions:

$$
\begin{aligned}
(1-q_i)\theta_i - b(\theta_i) - K_j &\geq 0 \quad (\text{if } \theta_i \geq \theta_j) \\
b(\theta_j) - K_j &\geq \theta_i q_i \quad (\text{if } \theta_i \leq \theta_j) \\
(1-q_j)\theta_j - b(\theta_j) + K_j &\geq 0 \quad (\text{if } \theta_i \leq \theta_j) \\
b(\theta_i) + K_j &\geq \theta_j q_j \quad (\text{if } \theta_i \geq \theta_j) \\
\int_{\underline{\theta}}^{\bar{\theta}} \tau F(\tau) dF(\tau) - \int_{\theta_j^*}^{\bar{\theta}} \tau dF(\tau) &\leq K_j \leq \int_{\theta_i^*}^{\bar{\theta}} \tau dF(\tau) - \int_{\underline{\theta}}^{\bar{\theta}} \tau F(\tau) dF(\tau)
\end{aligned}
$$

where K_j is possible transfer from i to j.

From the first two inequalities, the necessary and sufficient condition is that

$$(1-q_i)\theta_i - b(\theta_i) - K_j \geq 0 \ \& \ b(\theta_i) - K_j \geq \theta_i q_i$$

This requires $K_j \leq \frac{1}{2}(1-2q_i)\theta_i$. Looking at the third and fourth inequality, however, it requires $K_j \geq \frac{1}{2}(2q_j - 1)\theta_j = \frac{1}{2}(1-2q_i)\theta_j$. It is impossible to have $\frac{1}{2}(1-2q_i)\theta_j \leq K_j \leq \frac{1}{2}(1-2q_i)\theta_i$ for all θ_i and θ_j (as long as the type space is not trivially separating).

(ii) Suppose that i=S owns 1 unit of endowment, like a seller and we let i=B denote the buyer. For any incentive compatible payment rule $M(\theta_i, \theta_j)$, ex post individual rationality requires the following inequalities:

$$
\begin{aligned}
S(\theta_B, \theta_S) - M(\theta_B, \theta_S) - K_S &\geq 0 \quad (\text{if } \theta_B \leq \theta_S) \\
M(\theta_B, \theta_S) - K_S &\geq 0 \quad (\text{if } \theta_B \geq \theta_S) \\
S(\theta_S, \theta_B) - M(\theta_S, \theta_B) + K_S &\geq v(1, \theta_S) \quad (\text{if } \theta_B \geq \theta_S) \\
M(\theta_S, \theta_B) + K_S &\geq v(1, \theta_S) \quad (\text{if } \theta_B \leq \theta_S) \\
v(1, \bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial}{\partial \tau} v(x_S^*(\tau, \theta_j), \tau) d\tau dF(\theta_j) &\leq \frac{1}{2}\mathbb{E}\Delta + K_S \\
0 &\leq \frac{1}{2}\mathbb{E}\Delta - K_S
\end{aligned}
$$

Looking at the first two inequalities, it is necessary to have

$$K_S \leq \frac{1}{2} S(\underline{\theta}_S, \underline{\theta}_B)$$

since $K_S \leq \min\{S(\theta_B, \theta_B) - M(\theta_B, \theta_S), M(\theta_B, \theta_S)\}$.

Moreover, from the third and fourth inequality, we have

$$\frac{1}{2} S(\underline{\theta}_S, \underline{\theta}_B) \geq K_S \geq v(1, \bar{\theta}_S) - \frac{1}{2} S(\bar{\theta}_S, \bar{\theta}_S)$$

If $v(x, \bar{\theta}_S)$ is linear in x, then we go back to (i). If $v(x, \bar{\theta}_S)$ is strictly concave in x, then $v(1, \bar{\theta}_S) - \frac{1}{2} S(\bar{\theta}_S, \bar{\theta}_S) > 0$, a contradiction with $K_S \leq 0$. Q.E.D. ■

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