Continuously Dynamic Monopoly Pricing with Finite Horizon

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Abstract: This paper considers dynamic pricing strategy in a durable goods monopoly market with a finite time horizon. We compare the market openning has different frequencies and the frequency goes to infinity, which becomes continuously market openning. For a time consistent continuously dynamic pricing, the seller will get a positive expected profit which equals to the profit he can get when all transactions happen at the market-closing point; in addition, the profit does not related with buyers time preferences. Hence the Coase Conjecture is not true with finite time horizon. The Coase Conjecture is true if and only if the time horizon goes to infinity.

Key Words: the Coase Conjecture, Monopoly Pricing, Finite Horizon, Frequency

1 Introduction

Classical microeconomic theory suggests that supplier sells less in a steady monopoly market than in a competitive market, so monopoly leads to inefficiency. However, Coase guesses that monopoly does not lead to inefficiency all the time. More specially, in a repeated-opening durable good market, if the supplier is not able to commit to the future price, as the frequency of market opening rises, his power of getting monopoly profit diminishes gradually. And no extra profit will be gained by the supplier if the interval between market openning twice goes to zero. Such point of view was first given by Coase (1972).

About ten years after Coase (1972) was published, some economists began to analyze the above process with more precise mathematical framework and reached the following conclusion: in a durable monopoly market, once the interval between market opening twice goes to zero, all efficient business will happen in the first twinkling of an eye, and the price will be equal to the competitive price. Based on the conclusion above, it is thought that in a repeated opening durable good monopoly market, an efficient allocation can be realized if the supplier cannot commit to the future price credibly.

However, the above view depends on the assumption that market will open in the future in infinite time. It is worthwhile to note that the above conclusion that durable monopoly can be efficient also depends on the setting that the frequency of market opening must be high enough, here to be continuous. But once the market opens in infinite time, the frequency of market opening is not an important issue. Because the length of the interval does not have a substantial influence on this problem if the interval between market opening twice is finite while the market will open in infinite time. So, we consider that a study of the case that market opens only in a finite time interval is valuable. When the market opens only in a finite time, our study shows that the conclusion about market's efficiency does not hold any more.

Concretely, we find, on the one hand, when the market opens discretely in a given finite interval, the supplier's expected profit gradually decreases, but always keeps higher than a positive benchmark profit, as the frequency rises; on the other hand, once the market opens continuously, the supplier can expect only a benchmark profit which equals to the discount value of a one-period-monopoly profit at the market closing time. Interestingly, the above benchmark profit depends only on the time preference of the seller and it is independent of the buyers' time preference, although the price dynamic changes with different buyers' time preference. Not surprisingly, our study also shows that when the market opens discretely, buyers' time preference also influences seller's expected profit. If the buyers are more patient, the price dynamic is smoother; if the seller is more patient, price will decrease more.And our conclusion also tells the monopoly buyer's optimal strategies in the exhaustible resource market.

Our analysis also can be interpreted as a sequential bargaining game with deadline and private information. A preset deadline in a negotiation is not unusual in the real world. A company which faces refinancing deadline has to sell some share of stock to another, two parties need to compromise before the deadline given by the court, etc. All these negotiations can be treated as a sequential bargaining with deadlines. We analysis the sequential bargaining in which the buyer holds private information and the seller can make a take-it-or-leave-it offer, and show an explicit solution of such game with preset deadline.

If the buyers in our model can get a revenue flow from the durable good, rather than a quantity, our analysis still works and our main conclusion also holds.

The remainder of this paper is organized as follows. In Section 2, a literature review about Coase Conjecture and related topics are displayed. In Section 3, we introduce the model setup. In Section 4, a uniform case is analyzed with discrete setting and an iterative solution is given. In section 5, we give a general analysis about the continuous framework, and depict some characteristics of the equilibrium. In Section 6, we give a comparative analysis to the uniform case equilibria with discrete setting and continuous setting, and discuss how frequency and time preference influence the equilibrium outcome. In Section 7, we conclude and suggest extensions to the future work.

2 Literature Review

Coase(1972) supposes there is a unique land supplier, who sells his land with the object of profit maximization instead of social welfare maximization. Then, as classical monopoly theory tells, when the market opens in the first time, a price higher than the competitive price level would be given and the trading volume would be smaller than the competitive volume. The problem is, after trading in the first time, the supplier still owns some land and has an incentive to sell the residual land with a lower price, so he can get some extra revenue. It can be predicted that after infinite periods of trading, all land will be sold. However, once the start to expect that price will get down in the future, it is not surprising that some of them will choose to wait for a lower price instead of accepting the first higher price. As the supplier cannot commit to the future price, he will not get the monopoly profit. With such logic, Coase Conjecture is given as follows, if the interval between market opening twice goes to zero, two outcomes will happen, the one is that all land will be traded in the first twinkling of an eye, and the other is that the price will converge to the competitive price level.

Stokey (1981), Sobel and Takahashi (1983), Gul, Sonnenschein and Wilson (1986), and many others have given more precise mathematical analysis on this topic. Their studies show that, Coase Conjecture is quite penetrating; it points out that how is the bargaining power of both the buyer and the seller influenced by incredible commitment about the future price in a dynamic game framework with a clear logic. Stokey(1981) characterizes some properties of rational expected equilibrium (REE) and perfect rational expected equilibrium (PREE) in a durable good monopoly. She points out that buyers' expectation must be fulfilled along the realized path of production ,but it gives no extra constraint to the equilibrium, and any profile of buyers' strategies can compose an equilibrium; PREE only needs buyers' strategies to be right-continuous, then it also does not exist uniquely. Only if expectations depend continuously on the current stock should the equilibrium be unique. The unique equilibrium is the one guessed by Coase (1972).

Sobel and Takahashi (1983) normalize the buyers on the unit interval, then interpret this model as an bargaining game in which the seller gives a take-it-or-leave-it offer, and the buyer decides to accept it or reject it based on his private information. This paper highlights the significant role of commitment in this game, using backward induction, and does some comparative static analysis to show how do the buyer and the seller's patience, the buyer's distribution and other factors influence the equilibrium.

Gul, Sonnenschein and Wilson (1986) give a normalized solution to the Coase Conjecture in infinite time, and clearly show the following two outcomes, (1) all business will happen in the first twinkling of an eye, (2) the price will converge to the competitive price level in the first period.

Following Gul, Sonnenschein and Wilson (1986),there are many papers about the Coase Conjecture and related topics. Jacques Thepot (1998) gives a direct mathematical proof to the Coase Conjecture. Gul (1987) studies dynamic pricing in durable goods oligopoly. Horner and Kamien (2004) point out that The Coase Conjecture and the theory of exhaustible resource pricing mirror each other perfectly. Deneckere (2008)studies the Imperfect durable goods monopoly and compares it with the Coase Conjecture. Ausubel and

Deneckere(1989) restudies the multiple equilibria issues and shows that besides the Coase Conjecture there are still infinite equilibria in a durable goods monopoly market.Dudine, Hendel, and Lizzeri (2006)

Above literature all focus on the infinite horizon case, and their conclusions uniformly support the view that the durable monopoly market can be efficient. Our paper analyzes this topic with the basic model setup similar Sobel and Takahashi (1983)and many other literature such as Bulow(1982), characterizes the equilibria of finite horizon case, which contains both discrete iterative solution and continuous explicit solution of uniform case, under the condition that seller cannot commit to the future price. Our study shows that classic literature's conclusion, market is efficient, does not hold if market opens only in finite time. However, Coase's logic still holds, the supplier can get only a benchmark profit. And we also show that the classic view that monopoly can get no profit with infinite horizon is a special case of our outcome.

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3 Model Setup

Suppose there is one seller with a durable good. The seller's discount rate is r_S. There are heterogeneous buyers with different reservation value , and the buyers' discount rate is r_B. Without loss of generality, suppose the cumulative distribution function of v is $F : [0,1] \to [0,1]$. Denote as the PDF and we assume that $f(v) > 0$ for all $v \in [0,1]$. Trade happens repeatedly. The length of a period is given by Δt. Suppose there are total $n+1$ periods, and let $t = n\Delta t$.

Time is discrete and there are in total $N+1$ periods. Each period has length Δ , and let $T = n\Delta$ be the total length of market opening time. In each period $m \in \{0, 1, \ldots, N\}$, the seller sets a price, $p_m \in [0,1]$, and the buyers choose to accept it or to wait. If a buyer with reservation value v accepts price p_m, his payoff is given by $e^{-r_B m\Delta}(v - p_m)$, and if a buyer does not accept any price, his payoff is zero. At time $t = m\Delta$, if the upper bound value of the seller's who have not yet accept the price is q_m , the total payoff of the seller is $\sum_{i=0}^{N} e^{-r_S i\Delta} p_i(F(q_{i-1}) - F(q_i))$, where $q_{-1} = 1$.

At the beginning of each period , a history is given by a sequence of past prices $p^{m-1} \equiv (p_0, \ldots, p_{m-1})$. As usual, the history at the beginning of period 0 is given by the empty set. The well-known skimming property states that in any sequential equilibrium, following any history p^{m-1} and current price p_m, there exist a cutoff level $\kappa(p^{m-1}, p_m)$ such that the remaining buyers accept the offer if and only if their reservation values exceed $\kappa(p^{m-1}, p_m)$.

In our analysis, we restrict the buyer and the seller to use Markovian strategies. The skimming property implies that the buyer and the sellers' strategies depend only on the cutoff level and the current time. In particular, if the cutoff level is at the current time t (i.e., $t = m\Delta$ for some $m \in \{0, \ldots, N\}$), the seller charges a price given by $p(\kappa, t)$, the buyers accept the price if and only if $v \geq k(p, \kappa, t)$.

Definition 1: A strategy profile (p^*, k^*) is a Markovian Equilibrium if the following three conditions hold.

(1) If the buyers' strategy is given by k^*, the seller maximizes his expected discounted

payoff by choosing price $p = p^*(\kappa, t)$;

(2) If the seller's strategy is given by p^*, and the sell whose reservation value is v, he maximizes his expected discounted payoff by choosing accept p if and only if $v \geq k^*(p, \kappa, t)$;

(3) The cutoff level belief κ evolves according to the past history of plays.

The evolution of the cutoff beliefs is not restricted to equilibrium play alone. In particular, if the cutoff level is κ at time t, then the cutoff level at time $t + \Delta t$ is given by $min\{k^*(p, \kappa, t), \kappa\}$ for all chosen by the seller. If both the seller and the buyers follow the equilibrium strategy, denote $K(s, \kappa, t)$ as the equilibrium cutoff level at time $s \geq t$.Note that $K(s, \kappa, t) = \kappa$.

4 Discretely Backward Induction: An Iterative Solution of the Uniform Case

We solve the above game by backward induction with a set of uniformly distributed buyers first. Then, $F(v) = v, f(v) = 1, \forall v \in [0, 1]$. At time T, suppose the cutoff level is q_{N-1}, the seller charges a price p_N. Since a buyer's payoff is zero if he refuses, he accepts the given price if and only if his value exceeds the offered price. This implies that $q_N = p_N$. And the seller's payoff is $\pi_N = p_N(q_{N-1} - q_N)$.

$$p_N = \arg\max_{p_N} p_N(q_{N-1} - q_N)$$

$$s.t. q_N = p_N$$

This implies that $p_N = \frac{1}{2}q_{N-1}$, and $\pi_N = \frac{1}{4}q_{N-1}^2$. For any $m \in \{0, 1, ..., N-1\}$, we can write the seller's expected payoff as follows:

$$\pi_m = \max_{p_m} p_m(q_{m-1} - q_m) + e^{-r_S \Delta}\pi_{m+1} \tag{1}$$

$$s.t. q_m - p_m = e^{-r_b \Delta}(q_m - p_{m+1}) \tag{2}$$

Suppose $p_i = k_i q_{i-1}, q_i = g_i q_{i-1}, \pi_i(q_{i-1}) = h_i q_{i-1}^2, i \in \{0, 1, ..., N\}$, as above, we can get

$$k_N = \frac{1}{2}, g_N = \frac{1}{2}, h_N = \frac{1}{4}$$

With $\{k_j, g_j, h_j\}_{j=i+1}^{N}$ given, from (2), we can get $g_i - k_i = e^{-r_b\Delta}(g_i - k_{i+1}g_i)$, so

$$k_i = g_i(1 - e^{-r_b\Delta}(1 - k_{i+1})) \tag{3}$$

From (1), we can get

$$\pi_i = \max_{k_i} k_i q_{i-1}(q_{i-1} - g_i q_{i-1}) + e^{-r_S\Delta} h_{i+1}(g_i q_{i-1})^2 \tag{4}$$

Combining (3) and (4), we have

$$\pi_i = \max_{g_i} g_i(1 - e^{-r_b\Delta}(1 - k_{i+1}))q_{i-1}(q_{i-1} - g_i q_{i-1}) + e^{-r_S\Delta} h_{i+1}(g_i q_{i-1})^2 \tag{5}$$

FOC:$(1 - e^{-r_b\Delta}(1 - k_{i+1}))(1 - 2g_i) + 2e^{-r_S\Delta} h_{i+1} g_i = 0$

And we can derive

$$g_i = \frac{1 - e^{-r_b\Delta}(1 - k_{i+1})}{2(1 - e^{-r_b\Delta}(1 - k_{i+1})) - 2e^{-r_S\Delta} h_{i+1}} \tag{6}$$

So $\pi_i = [g_i(1 - e^{-r_b\Delta}(1 - k_{i+1}))(1 - g_i) + e^{-r_S\Delta} h_{i+1} g_i^2] q_{i-1}^2$, which means

$$h_i = g_i(1 - e^{-r_b\Delta}(1 - k_{i+1}))(1 - g_i) + e^{-r_S\Delta} h_{i+1} g_i^2 \tag{7}$$

Above all, we get the iterative solution of uniform case as follows.

Theorem 1: The iterative solution of uniform case is given as below:

$$\begin{aligned}
&p_i = k_i q_{i-1}, q_i = g_i q_{i-1}, \pi_i(q_{i-1}) = h_i q_{i-1}^2, i \in \{0, 1, ..., N\} \\
&k_N = \frac{1}{2}, g_N = \frac{1}{2}, h_N = \frac{1}{4} \\
&g_i = \frac{1 - e^{-r_b\Delta}(1 - k_{i+1})}{2(1 - e^{-r_b\Delta}(1 - k_{i+1})) - 2e^{-r_S\Delta} h_{i+1}} \\
&k_i = g_i(1 - e^{-r_b\Delta}(1 - k_{i+1})) \\
&h_i = g_i(1 - e^{-r_b\Delta}(1 - k_{i+1}))(1 - g_i) + e^{-r_S\Delta} h_{i+1} g_i^2 \\
&q_{-1} = 1
\end{aligned}$$

5 General Analysis and Explicit Solution of the Continuous Uniform Case

5.1 Discrete Analysis

We analyze the game using backward induction. Suppose the cutoff level is κ at time $t = T$. Since a buyer's payoff is zero if he refuses, he accepts the given price if and only if his value exceeds the offered price. This implies that $k^*(p, \kappa, T) = p$. Anticipating the buyers' action, the seller chooses a price to maximize his expected current-discounted payoff $\Pi(\kappa, t) \equiv \max_p p(F(\kappa) - F(p))$.

We assume that F is well-behaved so that the maximization problem has a unique solution. Define $g(\kappa) = \arg\max_p p(F(\kappa) - F(p))$. This implies that

$$p^*(\kappa, T) = g(\kappa), \Pi(\kappa, T) = g(\kappa)(F(\kappa) - F(g(\kappa)))$$

In addition, for a remaining buyer with value v, his payoff is given by

$$U(v, \kappa, T) = \max\{v - g(\kappa), 0\}$$

Now suppose p^*, k^*, Π and U are known for all $\{t + \Delta, t + 2\Delta, ..., T\}$. Their value at time t can be calculated as follows. First, at time t, suppose the current cutoff level is κ and the seller sets a price $p \in [0, \kappa]$. The seller's belief of the next period cutoff level is given by $k^*(p, \kappa, t)$. For a buyer of value v, his current-discounted payoff is then given by

$$U(v, \kappa, t) = \max\{v - p, e^{-r_b\Delta}U(v, k^*(p, \kappa, t), t + \Delta)\}$$

This determines U at t. Second, the skimming property implies that the buyer with reservation value equals to the cutoff level is indifferent from accepting today and accepting tomorrow (given seller follows the equilibrium strategy). In other words, the cutoff level is given by

$$k^*(p, \kappa, t) - p = e^{-r_b\Delta}(k^*(p, \kappa, t) - p^*(k^*(p, \kappa, t), t + \Delta)) \qquad \text{(Buyer's IC)}$$

When p^* and k^* are known at $t + \Delta$, the solution to the equation above determines $k^*(p, \kappa, t)$. Third, the buyer's incentive compatibility condition above implies that to obtain a cutoff level of x next period, the seller chooses a price

$$p = x - e^{-r_b\Delta}(x - p^*(x, t+\Delta))$$

This condition allows us to write the seller's value function recursively as

$$\Pi(\kappa, t) = \max_x (x - e^{-r_b\Delta}(x - p^*(x, t+\Delta)))(F(\kappa) - F(x)) + e^{-r_s\Delta}\Pi(x, t+\Delta) \quad \text{(Bellman)}$$

From the FOC with respect to x, we have

$$\begin{aligned} &(1 - e^{-r_b\Delta}(1 - p_1^*(x, t+\Delta)))(F(\kappa) - F(x^*)) \\ &-f(x^*)(x^* - e^{-r_b\Delta}(x^* - p^*(x^*, t+\Delta))) + e^{-r_s\Delta}\Pi_1(x^*, t+\Delta) = 0 \end{aligned} \quad \text{(FOC on Bellman)}$$

where the subscripts denote partial derivatives. When the solution to the FOC is unique, it helps solve for both p and Π at t. And this finishes the backward induction.

When $F(x) = x^\alpha$ for $\alpha > 0$, the equilibrium is unique and explicit formula for the equilibrium strategies are given in Sobel and Takahashi (1983) . However, for more general type distributions, the explicit solutions of equilibrium strategies are difficult to find and it is also not easy to find general conditions that guarantee the uniqueness of equilibrium. However, it is easy to give a lower bound to the seller's equilibrium expected profit.

Proposition 1: The seller's expected profit in any equilibrium is at least $e^{-r_sT}\Pi^s(1)$, where $\Pi^s(1) = g(1)(1 - F(g(1)))$ is the seller's monopolist's profit in the static setting.

This simple result is based on the observation that the seller cannot be made worse than waiting till the last period and then charge the monopolist price. This result is included here because this lower bound is also the limit of the seller's equilibrium profit as the length of each period goes to zero.

5.2 Continuous Analysis

To derive the continuous-time strategy, we take the limit of discrete-time equilibrium strategies as the length of the period Δ goes to zero. We are interested in equilibrium strategies that satisfy the following smooth assumptions.

Smoothness

(1)$p^*(\kappa,t)$ is differentiable in both argument for all (κ,t) in $(0,1)\times(0,T)$ and is continuous in $[0,1]\times[0,T]$;

(2)For each (κ,t) in $[0,1]\times[0,T)$, the equilibrium cutoff level $K(s,\kappa,t)$ satisfies that $K_1(s,\kappa,t)$ exists for all $s\in[t,T)$.

The focus on such "smooth" equilibrium seems natural on economic grounds and is common in the literature; see for example, Stokey (1981). In an infinitely-durable-goods monopolist setting, Stokey(1981) shows that any quantity path can be supported as a rational expectation equilibrium when the expectation does not need to be continuous.

Note the Smoothness assumption does not require the continuity of K at $t=T$. In other words, trade can occur in the last period with positive quantity. The positive quantity of trade in the last period appears consistent with the many last-minute selling in the real world. In addition, for the class of distribution $F(x)=x^\alpha$ with $\alpha>0$, the limit of discrete-time equilibrium also displays a positive quantity of trade in the last period; see Figure 2 in Appendix.

Our main results show that the equilibrium strategies take an extremely simple form.

Theorem 2: Under the Smoothness Assumption, the followings hold.

(1) $p^*(\kappa,t)=e^{r_b(t-T)}g(\kappa)$(Equilibrium price)

(2) $\Pi(\kappa,t)=e^{r_s(t-T)}\Pi^s(\kappa)$(Equilibrium Profit) where $\Pi^s(\kappa)=g(\kappa)(F(\kappa)-F(g(\kappa)))$ is the static monopolist profit with cutoff level κ.

Note that the profit here is equal to the profit given in Proposition 1, in which the seller only sells at the deadline. This outcome tells that monopoly cannot gain any extra profit when the market opens continuously. The intuition interpretation of such point is as follows: under equilibrium condition, if the seller can expect a higher profit at a point, the seller can make an improving by raising the price at the former time and getting a larger cutoff at the latter time. As we know, the interval between market opening twice goes to zero, the discount factor can be ignored in such interval, so the change leads to an improvement for the seller, but we know all improvement has been made because the seller has set an optimal price combination.

Proof: To prove (1), we show that the equilibrium price strategy satisfies a partial differential equation (PDE) and $p^*(\kappa, t) = e^{r_b(t-T)}g(\kappa)$ is its unique solution. The envelope condition of the seller's value function (from Bellman Equation above) implies that $\Pi_1(\kappa, t) = p^*(\kappa, t)f(\kappa)$. Using the envelope condition and substituting for in the FOC, we have

$$(1 - e^{-r_b\Delta}(1 - p_1^*(K, t + \Delta)))\frac{(F(\kappa)-F(K))}{f(K)}$$
$$-(K - e^{-r_b\Delta}(K - p^*(K, t + \Delta))) + e^{-r_s\Delta}p^*(K, t + \Delta) = 0$$

Where we use $K = K(t + \Delta, \kappa, t)$ to simplify notation.

Since p^* and K are both continuously differentiable, for small enough Δ, we have the following.

$$(1 - e^{-r_b\Delta}(1 - p_1^*(K, t + \Delta)))\frac{(F(\kappa)-F(K))}{f(K)}$$
$$= -p_1^*(K, t + \Delta)K_1(t + \Delta, \kappa, t)\Delta + o(\Delta)$$

$$K - e^{-r_b\Delta}(K - p^*(K, t + \Delta))$$
$$= p^*(K, t + \Delta) + r_b(K - p^*(K, t + \Delta))\Delta + o(\Delta)$$
$$e^{-r_s\Delta}p^*(K, t + \Delta) = p^*(K, t + \Delta) - r_s p^*(K, t + \Delta)\Delta + o(\Delta)$$

Combining the three parts, we have

$$(-p_1^*(K, t + \Delta)K_1(t + \Delta, \kappa, t) - r_b(K - p^*(K, t + \Delta)) - r_s p^*(K, t + \Delta))\Delta + o(\Delta) = 0$$

Since the strategies are Markovian, we have

$$K_1(t + \Delta, \kappa, t) = K_1(t + \Delta, K(t + \Delta, \kappa, t), t + \Delta)$$

Sending Δ to zero and using the continuity of K, we have

$$-p_1^*(\kappa, t)K_1(t, \kappa, t) - r_b\kappa + (r_b - r_s)p^*(\kappa, t) = 0 \qquad \text{(PDE-1)}$$

To determine $K_1(t, \kappa, t)$, we not that the cutoff level evolves according to the following:

$$K(t + \Delta, \kappa, t) - p^*(\kappa, t) = e^{-r_b\Delta}(K(t + \Delta, \kappa, t) - p^*(K(t + \Delta, \kappa, t), t + \Delta))$$

Send Δ to zero, the above equation simplifies to the following

$$r_b(p^*(\kappa, t) - \kappa) = p_1^*(\kappa, t)K_1(t, \kappa, t) + p_2^*(\kappa, t) \qquad \text{(PDE-2)}$$

Note this PDE-2 has a simple explanation, the left hand side measures the flow value of accepting the offer this period; the right hand side measures the price drop between the two periods. They must equal for the cutoff type.

Combining PDE-1 and PDE-2, we obtain the following PDE: for all $(\kappa, t) \in (0,1) \times (0,T)$,

$$r_s p^*(\kappa, t) = p_2^*(\kappa, t) \tag{PDE}$$

The continuity of p^* gives the boundary condition that $p^*(\kappa, T) = g(\kappa)$, where recall $g(\kappa)$ is the monopolist's choice of price in the static setting with cutoff level .

This PDE can be solved by separation of variables, and it is easy to check that

$$p^*(\kappa, t) = e^{r_b(t-T)} g(\kappa)$$

is a solution. Moreover, this solution is unique. To see this, suppose p^1 and p^2 are two different solutions to the PDE, then $p^d = p^1 - p^2$ also satisfies the PDE with the boundary condition at $t = T$ given by $p^d(\kappa, T) = 0$ for all κ. Now suppose $p^d(\kappa, t) > 0$ for some $\kappa > 0$, then the PDE implies that $p^d(\kappa, t') > 0$ for all $t' > t$(since $p_2^d(\kappa, t) = r_s p^d(\kappa, t) > 0$), this contradict the boundary condition at $t = T$. Similarly, we can show that it is impossible for $p^d(\kappa, t') < 0$ for any $\kappa > 0$. This implies $p^1 \equiv p^2$, and, thus, (1) holds.

To prove (2), recall from the envelope condition in the static monopolist problem implies that

$$\frac{d\Pi^s(\kappa)}{d\kappa} = g(\kappa) f(\kappa)$$

Since for all $t \in [0, T]$ we have $\Pi_1(0, t) = 0 = \Pi^s(0)$, (2) follows.

Next, let's study the price and quantity dynamics. Denote $K^*(t) = K(t, 1, 0)$ as the equilibrium quantity at time t. The initial condition of K^* is given by $K^*(0) = 1$.

From PDE-1, we have

$$-e^{r_b(t-T)} g'(K^*(t)) K^{*'}(t) - r_b K^*(t) + (r_b - r_s) e^{r_b(t-T)} g(K^*(t)) = 0$$

The solution to this equation depends on , which is typically a nonlinear function.

When $F(q) = q^m$, we have $g(q) = (\frac{1}{m+1})^{1/m} q$, and the equation above becomes

$$-e^{r_b(t-T)} (\frac{1}{m+1})^{1/m} K^{*'}(t) - r_b K^*(t) + (r_b - r_s) e^{r_b(t-T)} (\frac{1}{m+1})^{1/m} K^*(t) = 0$$

This is a first order linear ODE and has an explicit solution:

$$K^*(t) = \exp((r_b - r_s)t + (\frac{1}{m+1})^{-1/m} \frac{r_b}{r_s}(e^{r_s(T-t)} - e^{r_s T}))$$

Note that while the equilibrium profit depends only on the seller's discount rate, the equilibrium quantity depends on both sides' discount rate. In particular, we have

$$-\frac{K^{*\prime}(t)}{K^*(t)} = ((\frac{1}{m+1})^{-1/m} e^{r_s(T-t)} - 1)r_b + r_s$$

So the equilibrium quantity drops at a faster rate if the buyers are less patient.

Finally, the equilibrium price at time t, define $p^*(s, q, t)$ as the equilibrium price charged by the seller at time $s \geq t$ when the cutoff at time t is q . By the equilibrium price equation, we have $p^*(t, 1, 0) = e^{r_s(t-T)} g(K(t))$. When $F(q) = q^m$, the above analysis implies that

$$\begin{aligned} & p^*(t, 1, 0) = e^{r_s(t-T)}(\tfrac{1}{1+m})^{1/m} K(t) \\ & = (\tfrac{1}{1+m})^{1/m} \exp(r_b t - r_s T + (\tfrac{1}{m+1})^{-1/m} \tfrac{r_b}{r_s}(e^{r_s(T-t)} - e^{r_s T})). \end{aligned}$$

When $m = 1$, we have

$$p^*(t, 1, 0) = (\tfrac{1}{2} \exp(r_b t - r_s T + \tfrac{2r_b}{r_s}(e^{r_s(T-t)} - e^{r_s T})).$$

In particular, the equilibrium price at time t is lower for less patient buyers.

Corollary 1: The explicit solution of the uniform case is:

$$p^*(t, 1, 0) = (\tfrac{1}{2} \exp(r_b t - r_s T + \tfrac{2r_b}{r_s}(e^{r_s(T-t)} - e^{r_s T})).$$

This corollary gives the solution of the uniform case which is corresponding to the discrete iterative solution given in Section 4. And in Section 6 we will show that such solution is the limit of the discrete case while N tends to be infinite.

6 Price Discrimination and Incentive Compatibility

We have solved the equilibrium of uniform case in Section 4 and Section 5. Here are some numerical analyses to discuss how the parameter influence equilibrium outcome. All following numerical analysis are based on Theorem 1 and Corollary 1 of Theorem 2. Without loss of generality, let $T = 1$. Then, in the discrete case, when the market opens $N + 1$ periods, the discount factor between adjacent two periods is $e^{-r/N}$.

6.1 The equilibrium strategies of seller and buyer in finite horizon

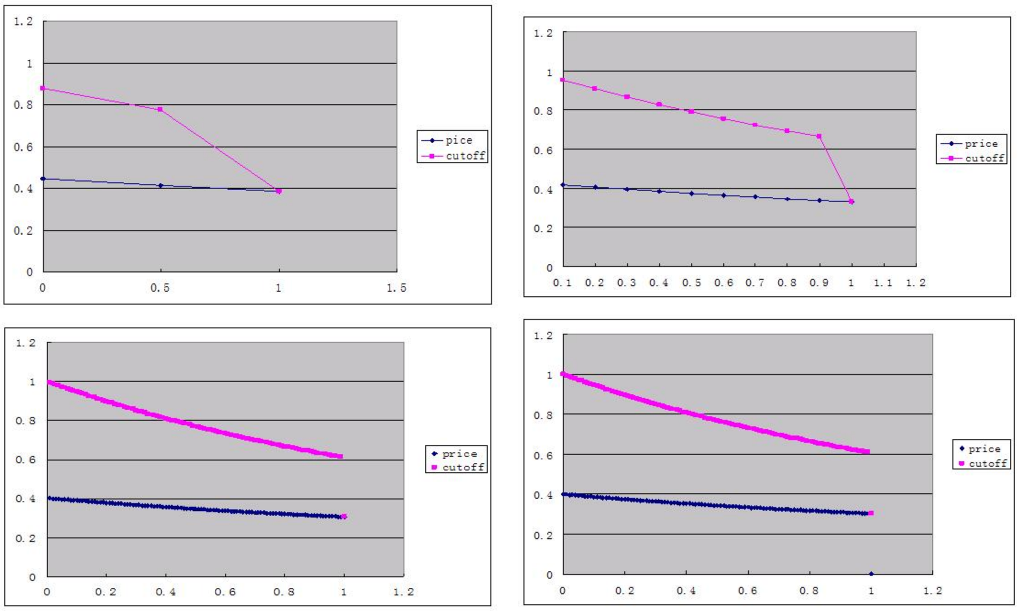

Figure 1: Equilibrium outcomes with different frequencies

The four pictures in Figure 2 show the equilibrium strategies of seller and buyer when the market opens 3 times, 10 times, 100times, and the market opens continuously with $e^{-r_s} = e^{-r_b} = 0.8$. From Figure 2 we can see that the continuous case is the limitation of the concrete case.

The four pictures in Figure 3 show the four continuous cases $e^{-r_s} = e^{-r_b} = 0.9$, $e^{-r_s} = 0.1, e^{-r_b} = 0.9$, $e^{-r_s} = 0.9, e^{-r_b} = 0.1$ and $e^{-r_s} = e^{-r_b} = 0.1$. We can see that more patient buyers lead to a smoother price series, as the former two pictures show, because the cost for patient buyer to wait is lower than for the impatient one, then trade volume until the last period will be less and the seller's belief will update in smaller degree; and similarly impatient buyers lead to a significant reduce of price. The second picture and the third picture make an interesting contrast: the later shows a process of price discrimination and the former shows a pooling process.

The two pictures in Figure 4 show the equilibrium outcomes with $e^{-r_s} = 0.6, e^{-r_b} = 0.9$ and $e^{-r_s} = 0.6, e^{-r_b} = 0.1$. Although price reduces significantly in the former case, while

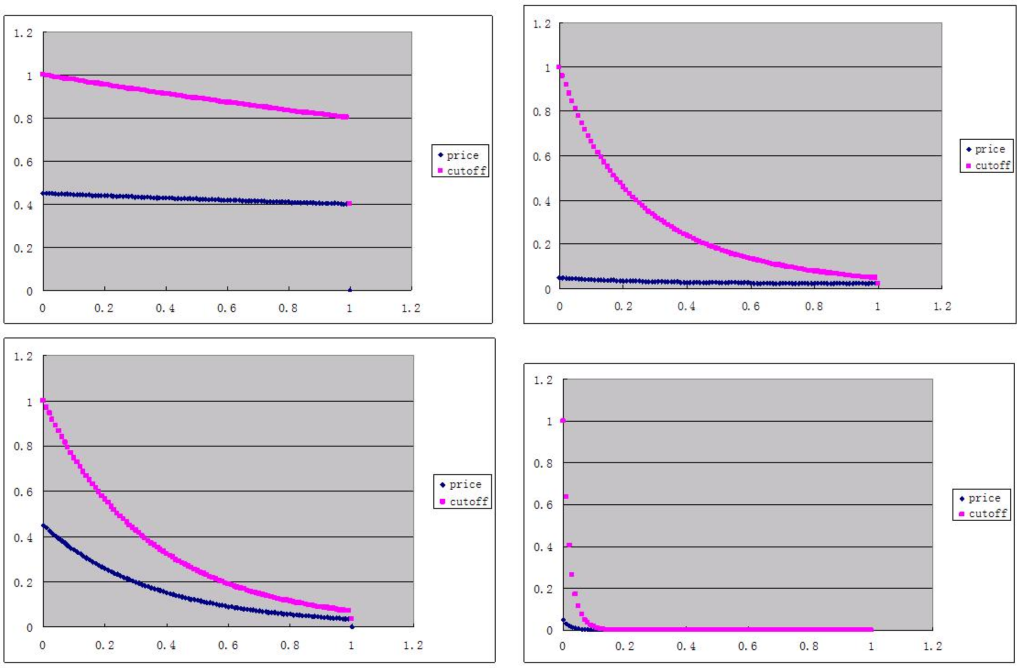

Figure 2: Equilibrium outcomes with different discount factors

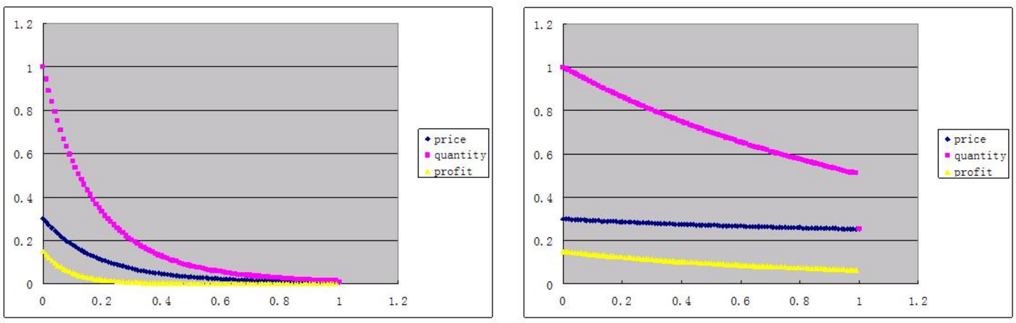

Figure 3: How buyer's discount factor influences?

in the latter case price changes smoother, we can see that seller's expected profit is the same. This same profit comes from the fact that both the seller and the buyers know that market will close at time T which equals to another fact that the seller can commit trading only in the time interval 0 to T.

6.2 The influence of frequency to seller's expected profit

The picture in Figure 5 shows the seller's expected profit with different frequencies when $e^{-r_s} = e^{-r_b} = 0.9$. By this picture we can see that as the frequency goes up, the seller's expected profit decreases and gets the lower bound when the market opens continuously.

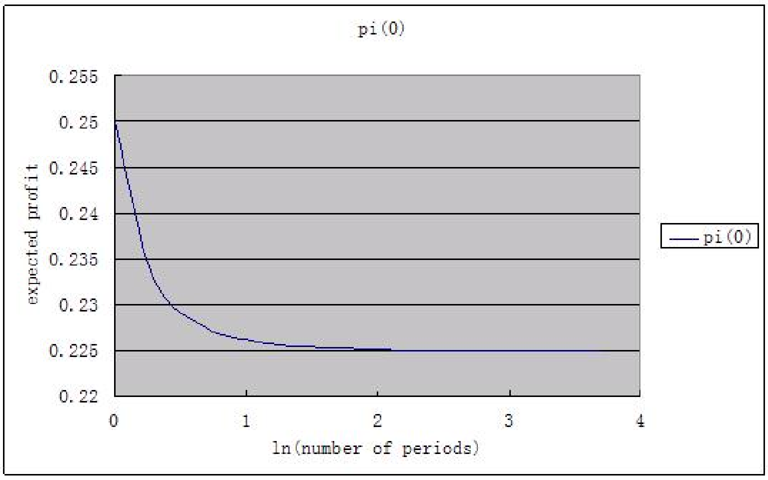

Figure 4: The seller's expected profit with different frequencies

6.3 The influence of frequency and both trader's patience to price and the seller's expected profit at $t = 0$

These two figures show the equilibrium price and seller's expected profit at $t = 0$ with $e^{-r_s} = 0.9$, and $e^{-r_b} = 0.99, 0.95, 0.9, 0.5, 0.1$. We can see that a lower frequency means a more significant influence of buyer's patience on equilibrium price and the seller's expected profit at $t = 0$.

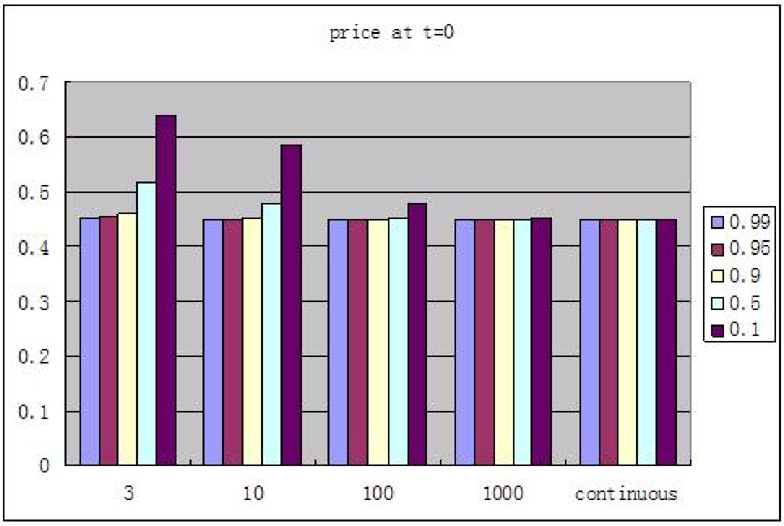

Figure 5: Price at $t = 0$ Figure 5 The seller's expected profit with different frequencies

Figure 6: The seller's expected profit at $t = 0$

7 Conclusion

This paper investigates the equilibrium of a finite horizon case in a durable good monopoly market and analyzes the factors which influence the equilibrium strategies when the seller cannot make a commitment about the future price. The influenced factors contains: the length of market opening time, market opening frequency and traders' time preference. Our studies show if the durable monopoly market opens in finite horizon, it cannot implement an efficient allocation, even if the market opens continuously. However, the seller in a continuously opening market cannot get a profit more than the profit getting by only selling at the market closing time. Such conclusion tells us, the view in Coase Conjecture that the seller cannot gain more through price discrimination because he cannot commit to the future price is right, while the corollary that the durable monopoly market is efficient does not hold.

More valuable study about this topic may contain: (1) what relation lies between the total social surplus and the length of total market opening time in this framework; (2) if the utility brought by the durable good is a flow instead of a quantity and the good is not perfect durable, how should the conclusion change; (3) if there exists heterogeneity about buyers' time preference, the conclusion above will hold, or not, etc.

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